CM2021/09/14

Tuesday, September 14, 2021 4:30 PM

First, we need to show that dyadic product is in fact a linear operator

$$(u \otimes v)(a + \beta b) \overset{?}{=} (u \otimes v)a + \beta (u \otimes v) b$$

$$(u \otimes v)(a + \beta b) = u \left(v \cdot (a + \beta b) \right) \quad \text{def.}$$

$$= u \left(v \cdot a + \beta \, v \cdot b \right) \quad \text{properties of inner product}$$

$$= u \underbrace{(v \cdot a)}_{\text{scalar}} + \beta \, u (v \cdot b)$$

$$= (u \otimes v) a + \beta (u \otimes v) b \quad \text{def. of } \otimes$$

So $u \otimes v$ is in fact a linear operator from $\mathcal{V} \to \mathcal{V}$

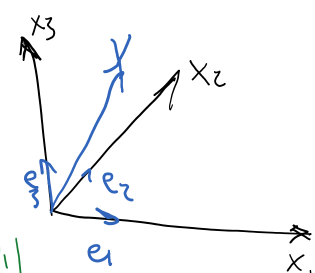
Expression of a 2nd order tensor in an orthonormal basis

Vectors basis $\{e_1, e_2, e_3\}$ e_i

$$v = v_i e_i$$

$$v_i = (v \cdot e_i) \quad \text{for orthonormal basis}$$

$$e_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad e_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$



$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

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basis? $e_i \otimes e_j$

$$T = T_{ij} \, e_i \otimes e_j \qquad T_{ij} = e_i \cdot T e_j$$

We want to show

$$T = T_{ij} \, e_i \otimes e_j \quad \text{where} \quad T_{ij} = e_i \cdot T e_j$$

All we need to do is to show for $\forall u$

$$Tu = (T_{ij} \, e_i \otimes e_j) u \quad \text{need to show this}$$

$$\underbrace{Tu}_{\text{vector}} = \underbrace{(Tu)_i}\, e_i \qquad v = Tu \quad \Big| \quad v = v_i e_i$$

$$\underbrace{Tu}_{\text{vektor}} = \underbrace{(Tu)_i}\, e_i$$

$$= (e_i \cdot Tu)\, e_i$$

$$= \Big[e_i \cdot \underbrace{T(u_j e_j)}\Big] e_i$$

$$= \Big[e_i \cdot \overbrace{[u_j\, Te_j]}\Big] e_i$$

$$= \Big[u_j \underbrace{[e_i \cdot Te_j]}_{\text{def. } T_{ij}}\Big] e_i$$

$$= T_{ij}\, \underbrace{u_j e_i}$$

$$= T_{ij}\, (e_i \otimes e_j)\, u$$

$$= [T_{ij}\, e_i \otimes e_j]\, u$$

$v = Tu$ $\begin{cases} v = v_i e_i \\ v_i = v \cdot e_i \end{cases}$ orthonormal basis

$u = u_j e_j$

$T(u_1 e_1 + \dots + u_n e_n)$

$= u_1 Te_1 + \dots + u_n Te_n$

T is a linear operator

$e_i \cdot (u_1 Te_1 + u_2 Te_2 + \dots)$

$u_1 e_i \cdot Te_1 + u_2\, e_i \cdot Te_2 \dots$

proporties of inner product

$a \cdot (\lambda b) = \lambda a \cdot b$

$a \cdot (b + c) = a \cdot b + a \cdot c$

$(e_i \otimes e_j)\, u =$

$e_i (e_j \cdot u) = e_i u_j$

$$\boxed{\begin{aligned} T &= (T_{ij}\, e_i \otimes e_j) \\ T_{ij} &= e_i \cdot Te_j \end{aligned}}$$

Similar to

$v = v_i e_i$

$v_i = v \cdot e_i$

$$e_1 \otimes e_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} [0 \;\; 1 \;\; 0] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e_2 \otimes e_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} [1 \;\; 0 \;\; 0] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we say

$$T = T_{11} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{e_1 \otimes e_1} + T_{12} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{e_1 \otimes e_2} + T_{13} \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{e_1 \otimes e_3}$$

$$T = T_{11}\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} + T_{12}\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix} + T_{13}\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$$

$$+ \cdots + T_{33}\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix} e_3 \otimes e_3 = \begin{bmatrix}T_{11}&T_{12}&T_{13}\\T_{21}&T_{22}&T_{23}\\T_{31}&T_{32}&T_{33}\end{bmatrix}$$

Basis for 3D, 2nd order tensor is

$$\underbrace{e_i \otimes e_j}_{\text{basis for 2nd order tensor}} = \left\{ \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \ldots, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix} \right\}$$

Theorem 51 *Let* $\mathbf{T} \in \operatorname{Lin} \mathcal{V}$ *with components* T_{ij} *w.r.t. the r.C.c.f.* X*, and let* $\mathbf{u}, \mathbf{v} \in \mathcal{V}$ *with components* u_i, v_i *w.r.t.* X*. Then*

$$\mathbf{Tu} = \mathbf{v} \Leftrightarrow T_{ij}u_j = v_i.$$

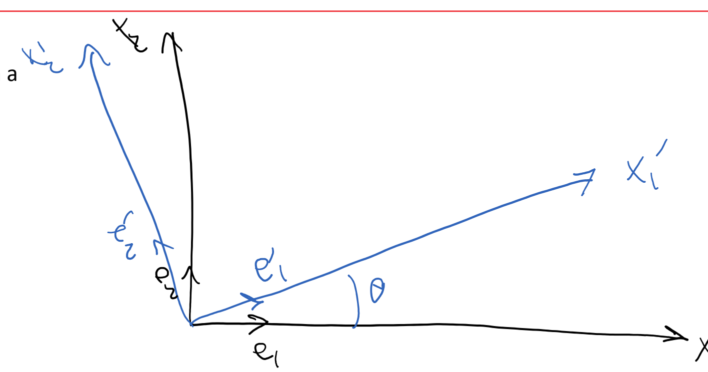
$v = Tu$

$v_i = T_{ij} u_j$

Proof. *Suppose* $\mathbf{Tu} = \mathbf{v}$*. Then*

$$\begin{aligned}
v_i &= \mathbf{e}_i \cdot \mathbf{v} \ \text{(components of } \mathbf{v}\text{)},\\
&= \mathbf{e}_i \cdot (\mathbf{Tu}) \ \text{(supposition)},\\
&= \mathbf{e}_i \cdot [(T_{kl}\mathbf{e}_k \otimes \mathbf{e}_l)\, \mathbf{u}] \ \text{(component representation of } \mathbf{T}\text{)},\\
&= \mathbf{e}_i \cdot [T_{kl}\,(\mathbf{e}_k \otimes \mathbf{e}_l)\, \mathbf{u}] \ \text{(defns. of tensor + and scalar mult.)},\\
&= \mathbf{e}_i \cdot [T_{kl}\,(\mathbf{e}_l \cdot \mathbf{u})\, \mathbf{e}_k] \ \text{(defn. of } \otimes \text{)},\\
&= \mathbf{e}_i \cdot [T_{kl}\,(u_l \mathbf{e}_k)] \ \text{(components of } \mathbf{u}\text{)},\\
&= T_{kl}u_l\,(\mathbf{e}_i \cdot \mathbf{e}_k) \ \text{(distr. and homog. of } \cdot \text{)},\\
&= T_{kl}u_l\delta_{ik} \ \text{(orthonormality of base vectors)},\\
&= T_{il}u_l = T_{ij}u_j \ \text{(property of } \delta_{ik} \text{ and labeling)}.\\
\therefore \mathbf{Tu} &= \mathbf{v} \Rightarrow T_{ij}u_j = v_i.
\end{aligned}$$

Matrix is the component expression of a 2nd order tensor in a particular coordinate system.



$v = v_i e_i$ $\quad \begin{bmatrix}v_1\\v_2\end{bmatrix}$

$v = v_i' e_i'$ $\quad \begin{bmatrix}v_1'\\v_2'\end{bmatrix}$

$v_i' = Q_{ij} v_j \qquad [v]' = Q[v]$

$v_j = Q_{ij} v_i' \qquad [v] = Q^T [v]'$

$Q = \begin{bmatrix} e_1' \\ \hline e_2' \end{bmatrix}$ expressed in e_1, e_2

$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$\underline{e_i' = Q_{ij} e_j}$

How about 2nd order tensors?

$$T = T_{ij}\, e_i \otimes e_j = T_{11}\, e_1 \otimes e_1 + \dots + T_{22}\, e_2 \otimes e_2 = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$ in non-prime system

$$T = T'_{ij}\, e'_i \otimes e'_j = \begin{bmatrix} T'_{11} & T'_{12} \\ T'_{21} & T'_{22} \end{bmatrix}$$ in prime system

how is $[T]$ related to $[T]'$

$$T = T_{ij}\, e_i \otimes e_j$$

$$\left(e'_m = Q_{mn} e_n \longleftrightarrow e_n = Q_{mn} e'_m \right)$$

$$e_i = Q_{mi} e'_m$$

$$e_j = Q_{nj} e'_n$$

$$T = T_{ij} (Q_{mi} e'_m) \otimes (Q_{nj} e'_n)$$

Theorem 46 *The tensor product is* homogeneous *and* distributive w.r.t. addition*:*

1. $(\alpha \mathbf{u}) \otimes (\beta \mathbf{v}) = \alpha\beta (\mathbf{u} \otimes \mathbf{v})\ \forall\, \mathbf{u}, \mathbf{v} \in \mathcal{V}$ and $\forall\, \alpha, \beta \in \Re$,
2. $\mathbf{u} \otimes (\mathbf{v} + \mathbf{w}) = \mathbf{u} \otimes \mathbf{v} + \mathbf{u} \otimes \mathbf{w}$ and $(\mathbf{u} + \mathbf{v}) \otimes \mathbf{w} = \mathbf{u} \otimes \mathbf{w} + \mathbf{v} \otimes \mathbf{w}\ \forall$ $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}$.

$$= T_{ij} Q_{mi} Q_{nj} (e'_m \otimes e'_n)$$

$$= Q_{mi} Q_{nj} T_{ij}\, e'_m \otimes e'_n$$

$$= T'_{mn}\, e'_m \otimes e'_n$$

$$\Longrightarrow \boxed{T'_{mn} = Q_{mi} Q_{nj} T_{ij}}$$

compare this to

$$v'_m = Q_{mi} v_i$$

another way to look at this

$$T'_{mn} = Q_{mi} T_{ij} Q^T_{jn} = (Q T Q^T)_{mn}$$

$$T_{mn} = Q_{mi} \underbrace{T_{ij} Q_{jn}} = (Q T Q^T)_{mn}$$

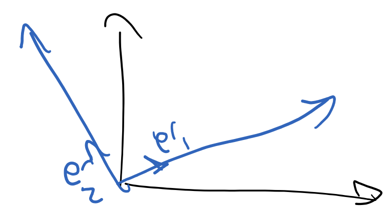
$$\boxed{[T]' = Q\,[T]\,Q^T}$$

another way to get here, but perhaps less rigorous ...

u, v vectors

$$v = Tu \qquad \overbrace{[v] = [T][u]}^{\text{components in ( ) system}} \quad (1) \qquad v_i = T_{ij} v_j$$

$$[v]' = \underbrace{?}_{[T]'} \; [u]'$$



$$[u]' = Q[u] \to [u] = Q^T[u]' \quad (2)$$

$$[v]' = Q[v] \to [v] = Q^T[v]' \quad (3)$$

plug (2) (3) in (1)

$$Q\left(Q^T[v]' = [T]Q^T[u]' \right)$$

$$\underbrace{QQ^T}_{I}[v]' = (Q[T]Q^T)[u]'$$

I because Q is orthonormal

$$[v]' = [T]'[u']$$

$$\downarrow$$

$$Q[T]Q^T$$

If T is a second order tensor, then its components are transferred between two coordinate systems as,

$$T'_{i_1 i_2} = Q_{i_1 j_1} Q_{i_2 j_2} \bar{T}_{j_1 j_2}$$

$$Q = \left[\frac{e'_1}{e_2} \; \vdots \right]$$

The inverse, if the components of 2 index array follow the coordinate transformation above, it's a 2nd order tensor

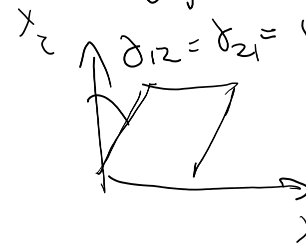
Theorem 53 *Let there be a set of nine real numbers associated with every r.C.c.f. in a three-dimensional Euclidean point space $\mathcal{E}$. For example, consider the sets $\{X, \mathbf{e}_i, T_{ij}\}$ and $\{X', \mathbf{e}'_i, T'_{ij}\}$ where X and X' are arbitrary frames with respective base vectors $\mathbf{e}_i$ and $\mathbf{e}'_i$. Then $\exists$ ("there exists") $\mathbf{T} \in \mathrm{Lin}\,\mathcal{V}$ given by $\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = T'_{ij}\mathbf{e}'_i \otimes \mathbf{e}'_j$ iff the two-index symbols T_{ij} and T'_{ij} satisfy the transformation rules*

$$T'_{ij} = \lambda_{ik}\lambda_{jl}T_{kl}; \quad T_{ij} = \lambda_{ki}\lambda_{lj}T'_{kl}.$$

for all choices of $\{X, \mathbf{e}_i, T_{ij}\}$ and $\{X', \mathbf{e}'_i, T'_{ij}\}$, where λ_{ij} are the cosines of the angles between $\mathbf{e}'_i$ and $\mathbf{e}_j$.

eng. shear strain

$\gamma_{12} = \gamma_{21} = u_{1,2} + u_{2,1}$



Example of a 2-array that is not a tensor

$\gamma_{11} = \varepsilon_{11} = \frac{\partial u_1}{\partial x_1}$

$\gamma_{22} = \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}$

$$\begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}$$

is not a tensor!

$$\begin{pmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{pmatrix}$$ is a 2nd order tensor

$$= \frac{\nabla u + \nabla u^T}{2}$$

$\epsilon_{12} = \epsilon_{21} = \frac{\gamma_{12}}{2}$

$= \frac{u_{1,2} + u_{2,1}}{2}$

Transpose of a tensor

Transpose of a second order tensor T is defined as,

$\forall u, v \in \mathcal{V} \quad v \cdot T^t u = u \cdot T v \qquad u \cdot T v = v \cdot T^t u$

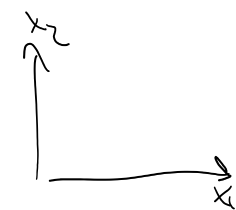
$\forall u, v \in \mathcal{V}) \quad v \cdot T^t u = u \cdot Tv \qquad u \cdot Tv = v \cdot T^t u$

$T^t u = ?$ in $e_1, \dots, e_n$ coordinate

$(T^t u)_i = (e_i \cdot T^t u) = u \cdot Te_i$

What can we say about T^t_{ij} in terms of components of T in a coordinate system?

T^t_{ij}



$(T^t)_{ij} = e_i \cdot [(T^t) e_j] \overset{\text{def of } T^t}{=} e_j \cdot T e_i = T_{ji}$

Some properties of transpose:

1) $(T^t)^t = T$

2) $(T + S)^t = T^t + S^t$

3) $(\alpha T)^t = \alpha T^t$

4) $I^t = I$

5) $(u \otimes v)^t = v \otimes u$

choose a coord system

$(u \otimes v)_{ij} = e_i \cdot [(u \otimes v) e_j] = e_i \cdot [u \, (\underbrace{v \cdot e_j}_{v_j})] = e_i \cdot [u \, v_j] =$

$= v_j \, (e_i \cdot u) = \underline{v_j \, u_i}$

$[(u \otimes v)^t]_{ji} = (u \otimes v)_{ij} = v_j u_i$

$(u \otimes v)^t \qquad (u \otimes \;)$

$$(u\otimes v)^t_{ji} = (u\otimes v)_{ij} = v_j u_i$$

$u \to v$

$v \to u$

$i \to j$

$j \to i$

$$(v\otimes u)_{ji} = u_i v_j$$

$$(u\otimes v)^t_{ji} = (v\otimes u)_{ji}$$

$$(u\otimes v)^t = v\otimes u$$

$$e_1\otimes e_2 = \begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0 & 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

$$e_2\otimes e_1 = \begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\1 & 0 & 0\\0 & 0 & 0\end{bmatrix} = (e_1\otimes e_2)^t$$

Some definitions related to symmetry:

Symmetric tensor:

$$T^t = T \qquad T_{ij} = T_{ji}$$

$$\mathrm{Sym} = \{T \in \underbrace{\mathrm{Lin}}_{\text{lin operators}} \mid T^t = T\}$$

Skew-symmetric $\quad T^t = -T$

$$\mathrm{Skew} = \{T \in \mathrm{Lin} \mid T^t = -T\}$$

Any 2nd order tensor can be written as the sum of a symmetric and skew symmetric tensors

$T = \underset{\vee}{\text{Sym}\,T} + \text{skew}\,T$

$\frac{T+T^t}{2} \qquad \frac{T-T^t}{2}$

side note

$H = \nabla u$

$H = \underbrace{\text{sym}\,H}_{\varepsilon} + \underbrace{\text{skew}\,H}_{W}$

small. def. rotation

Tensor product:

$S, T \in \text{Lin} \qquad ST \in \text{Lin} \qquad \text{where} \quad (ST)u := S(T(u))$

Some properties of tensor product:

1. $(RS)T = R(ST)$ associative
2. $T(R+S) = TR + TS$ distributive
3. $(\alpha R)(\beta T) = (\alpha\beta)RT$ homogeneity
4. $IT = TI = T$
5. $(RS)^t = S^t R^t$

Component expression $\quad (ST)_{ij} = ?$

$(ST)_{ij} = S_{ik} T_{kj}$

Theorem 61 *Let $\mathbf{S}, \mathbf{T} \in \text{Lin } \mathcal{V}$. The rectangular Cartesian components of the product $\mathbf{ST}$ w.r.t. a given r.C.c.f. $\{X, \mathbf{e}_i\}$ are*

$$(\mathbf{ST})_{ij} = S_{ik} T_{kj},$$

where S_{ik} and T_{kj} are the components of $\mathbf{S}$ and $\mathbf{T}$. w.r.t. X.

Proof. According to Definition 28, the components of $\mathbf{ST}$ w.r.t. the frame $\{X, \mathbf{e}_i\}$ are given by

$$\begin{aligned}
(\mathbf{ST})_{ij} &= \mathbf{e}_i \cdot (\mathbf{ST})\, \mathbf{e}_j \\
&= \mathbf{e}_i \cdot \mathbf{S}\,(\mathbf{Te}_j) \ \text{(definition of tensor product)}, \\
&= \mathbf{e}_i \cdot \mathbf{S}\,(\mathbf{Te}_j)_k\, \mathbf{e}_k \ \text{(component form of vector } \mathbf{Te}_j\text{)}, \\
&= \mathbf{e}_i \cdot \mathbf{S}\,[\mathbf{e}_k \cdot (\mathbf{Te}_j)]\, \mathbf{e}_k \ \text{(Theorem 36)}, \\
&= \mathbf{e}_i \cdot \mathbf{S}\,(T_{kj} \mathbf{e}_k) \ \text{(Definition 28)}, \\
&= \mathbf{e}_i \cdot (T_{kj} \mathbf{S} \mathbf{e}_k) \ \text{(linearity of } \mathbf{S}\text{)}, \\
&= T_{kj} \mathbf{e}_i \cdot \mathbf{S}\mathbf{e}_k \ \text{(linearity of inner product)}, \\
&= T_{kj} S_{ik} = S_{ik} T_{kj} \ \text{(Definition 28, properties of } \Re\text{)}. \ \blacksquare
\end{aligned}$$