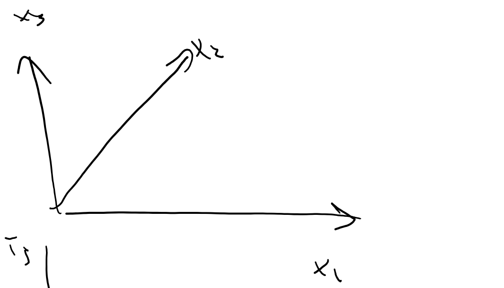
CM2021/09/16

Thursday, September 16, 2021 4:29 PM

Determinant of a second order tensor

Express the components of a tensor in a given orthonormal coordinate system:



$T \qquad u \to Tu$

$\{e_1, e_2, e_3\} \to T = T_{ij}\, e_i \otimes e_j \qquad [T]_e = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ & \ddots & T_{33} \end{bmatrix}$

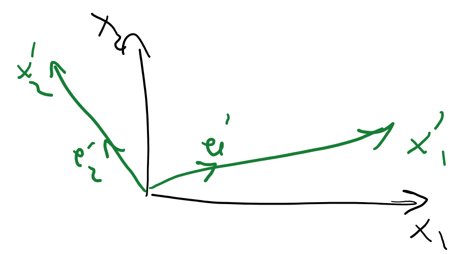
components in ( ) coordinate system

$\det T = \det [T_{ij}]$

matrix expression of T in this particular coordinate system

$\left\{ \frac{1}{6} \epsilon_{ijk}\, \epsilon_{pqr}\, A_{ip} A_{jq} A_{kr} \right.$

Since in this case, the definition is coordinate-system dependent, we need to show that the value of the determinant is scalar, meaning that we get the same number regardless of the coordinate system.



$\det [T_{ij}] \overset{?}{=} \det [T'_{ij}]$

$\det [T'_{ij}] = \frac{1}{6} \epsilon_{ijk}\, \epsilon_{pqr}\, T'_{ip} T'_{jq} T'_{kr}$

$T'_{ip} = Q_{ia} Q_{pd} T_{ad}$

$T'_{jq} = Q_{jb} Q_{qe} T_{be}$

$T'_{kr} = Q_{kc} Q_{rf} T_{cf}$

$\det [T'_{ij}] = \frac{1}{6} \epsilon_{ijk}\, \epsilon_{pqr} (Q_{ia} Q_{pd} T_{ad})(Q_{jb} Q_{qe} T_{be})(Q_{kc} Q_{rf} T_{cf})$

$= \frac{1}{6} (\epsilon_{ijk} Q_{ia} Q_{jb} Q_{kc})(\epsilon_{pqr} Q_{pd} Q_{qe} Q_{rf}) T_{ad} T_{be} T_{cf}$

$\frac{1}{6} \epsilon_{abc} (\det Q)^2 (\epsilon_{def} \det Q) \quad T_{ad} T_{be} T_{cf}$

$\epsilon_{abc}(\det Q)\,(\epsilon_{def} \det Q)\, T_{ad} T_{be} T_{cf}$

$= \det[T_{ij}]$

$QQ^t = I$

$\det QQ^t = 1 = \det Q\,(\det Q)^t = (\det Q)^2$

$\det Q^2 = 1$

Some properties of determinant:

$\det ST = \det S \det T$

$\det S^t = \det S$

$\det I = 1$

$\det 0 = 0$

$\det u \otimes v = 0$

$\det \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} [v_1 \;\; v_2 \;\; v_3]$

$\det \begin{bmatrix} v_1 \cdot \vec{v} \\ v_2 \cdot \vec{v} \\ v_3 \cdot \vec{v} \end{bmatrix} = 0$

Example

$\det e_1 \otimes e_2 = \det \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$

Trace of a second order tensor:

- Trace is defined through the following conditions:

1) Trace is a linear operator

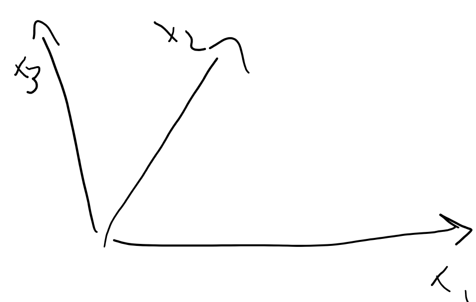
$tr(T + \alpha S) = tr(T) + \alpha\, tr(S)$ — α: scalar; T, S: 2nd order tensors

2) we define

$tr(\underbrace{u \otimes v}_{\text{2nd order tensor}}) = \underbrace{u \cdot v}_{\text{scalar}}$

The second condition defines trace for the smallest building blocks of tensors, e.g. basis for 2nd order tensors

Assume we choose a coordinate system



$T = T_{ij}\, e_i \otimes e_j$

$tr(T) = tr(T_{ij}\, e_i \otimes e_j)$

$= T_{ij}\, \underbrace{tr(e_i \otimes e_j)}$

$= T_{ij}\,(e_i \cdot e_j)$

1) Linearity of trace

2) $trace(u \otimes v) = u \cdot v$

$= T_{ij}\ \delta_{ij} = T_{ii}$

$\text{trace} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ & & \\ & & T_{33} \end{bmatrix} = T_{11} + T_{22} + T_{33}$ sum of diagonal

Do we need to prove that

trace$(T) = T_{ii}$ is coordinate-independent?

No; initial def. was coordinate independent

but if we wanted

$j \to i \qquad T'_{ij} = Q_{im} Q_{jn} T_{mn}$

$T'_{ii} = Q_{im} Q_{in} T_{mn} = [Q^T_{mi} Q_{in}] T_{mn}$

$= (\underbrace{Q^t Q}_{I})_{mn} T_{mn}$

$= \delta_{mn} T_{mn} = T_{mm}$ $\boxed{\text{End}}$

Some properties of trace:

1. $\text{tr}\, T^t = \text{tr}(T)$
2. $\text{tr}\,(ST) = \text{tr}\,(TS)$
3. $\text{tr}\,(I) = d$
4. $\text{tr}(0) = 0$

$\text{tr} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 3$

d dimension of space

($d = 2$ 2D
$= 3$ 3D)

Definition of inner product for 2nd order tensors (Def 34 in our course notes)

$u \cdot v$ in coordinate system

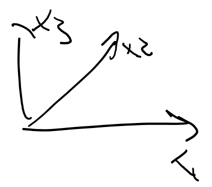
$u \cdot v = u_i v_i$

$T : S = T \cdot S$

$\begin{bmatrix} T_{11} & T_{12} & T_{13} \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \end{bmatrix}$

$$u \cdot v = u_i v_i$$

$$= u_1 v_1 + u_2 v_2 + u_3 v_3$$



$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & \rightarrow & \\ & & T_{33} \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} & \\ & & \\ & & S_{33} \end{bmatrix}$$

$$= T_{11} S_{11} + T_{12} S_{12} + \dots$$

$$= \underline{T_{ij} S_{ij}} \qquad T_{ij} S^t_{ji} = (TS^t)_{ii}$$

$$= \mathrm{trace}(TS^t)$$

Defines an inner product based norm:

$$\|T\| = \sqrt{T:T}$$

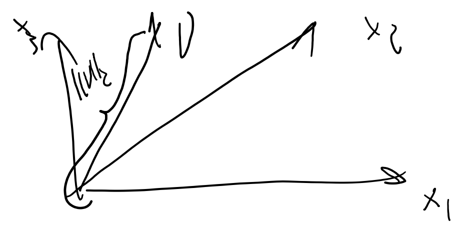
A very useful side note for you:

Vector-based norm for second order tensors

If we know how to define a norm for vectors, we can use that to define a norm for second order tensors:

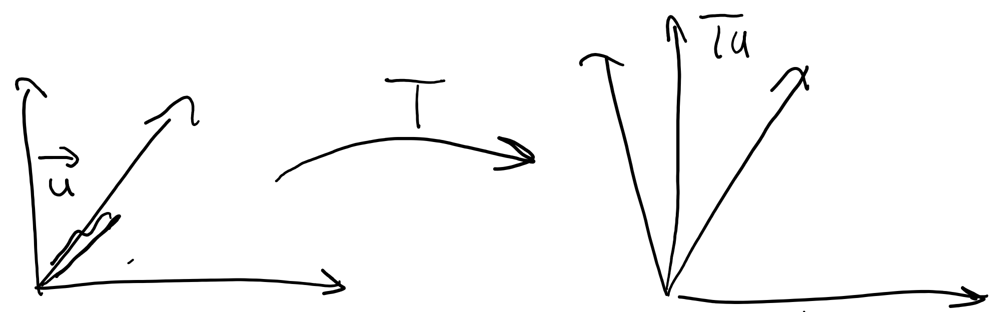
$$\|v\|_p = \sqrt[p]{|v_1|^p + |v_2|^p + |v_3|^p}$$

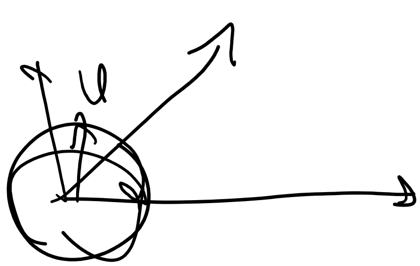
$$p = 2 \qquad \|v\|_2 = \sqrt{v_1^2 + v_2^2 + v_3^2}$$



T

$\|u\| = 1$



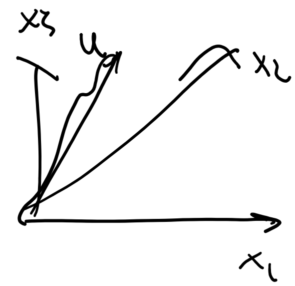


$$\|T\| = \max \underbrace{\|\overbrace{Tu}^{\text{vector}}\|}$$

$$\|T\| = \max_{\substack{u \neq 0 \\ \text{vector}}} \frac{\|Tu\|}{\|u\|}$$

vector-based (~ induced)

norm of a second order tensor



l_2 norm is the magnitude

l_2 vector based norm of a tensor is

$$\boxed{\max |\lambda_i| \quad \text{where } \lambda_i \text{ are eigenvalues of } T}$$

We can even define norm for 4th order elasticity tensor later

$$\underbrace{\sigma}_{\text{2nd order stress}} = \underbrace{C}_{\text{4th order elasticity tensor}} \underbrace{\varepsilon}_{\text{2nd order strain}}$$

$$\boxed{\|C\| = \max_{\varepsilon \neq 0} \frac{\|C\varepsilon\|}{\|\varepsilon\|}}$$

similar idea

or we could have defined $\|C\| = \sqrt{C:C} = \sqrt{C_{ijkl} C_{ijkl}}$

1.11.9 Inverse of a tensor

$$T^{-1} T = T T^{-1} = I$$

Theorem 76: Components of the inverse of T in a given coordinate system are:

$$T^{-1}_{ij} = \frac{1}{2\det T}\,\epsilon_{ipq}\,\epsilon_{jmn}\,T_{mp}\,T_{nq}$$

Inverse of a tensor exists if det T $\neq 0$

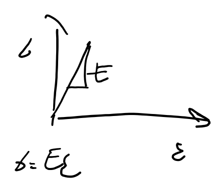
invertible tensors with >0 det $\quad \mathrm{Lin}^+ V = \{T \in \mathrm{Lin}\,V \mid \det T > 0\}$

$\mathrm{Inv}\,V = \{\ \cdot \mid \det \neq 0\}$

Higher order tensors

We already discussed elasticity tensor

$\sigma = C\,\varepsilon$ 2D, 3D generalization of 1D



tensor order $\quad \cancel{2} \quad 4 \quad 2$

Indicial notation

$\underbrace{v = Tu}_{\text{vectors}}$ 2nd order tensor

order = 2 - 1 (# contractions)

$v_i = T_{ij}\,v_j$

contracted

4th order

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{lk}$$

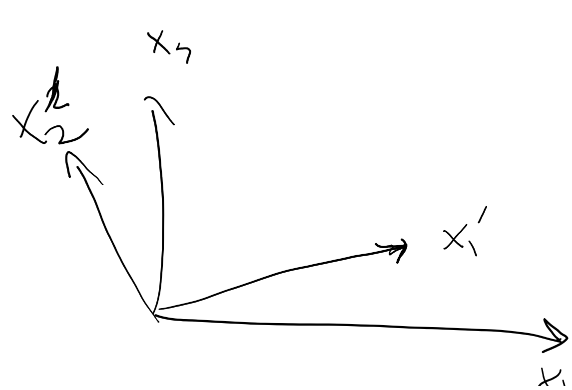
contraction #1

c. #2

Indicial notation of this tensor product

✓ $2 = 4 - 2$

Components of C in different coordinate systems:



$$C = C_{ijkl}\, e_i \otimes e_j \otimes e_k \otimes e_l$$

x_1

$$C = C_{ijkl}\, e_i \otimes e_j \otimes e_k \otimes e_l$$

$$\left.\begin{aligned} e_i &= Q_{mi}\, e'_m \\ e_j &= Q_{nj}\, e'_n \\ e_k &= Q_{ok}\, e'_o \\ e_l &= Q_{pl}\, e'_p \end{aligned}\right\} \rightarrow C = (Q_{mi} Q_{nj} Q_{ok} Q_{pl} C_{ijkl})\, e'_m \otimes e'_n \otimes e'_o \otimes e'_p$$

$$\boxed{C'_{mnop} = Q_{mi} Q_{nj} Q_{ok} Q_{pl} C_{ijkl}}$$

I should have first defined polyads used above ...
They are generalization of dyadic product:

$$(u_1 \otimes u_2)\, v = u_1 (u_2 \cdot v)$$

$$(u_1 \otimes \cdots u_{n-1} \otimes u_n)\, v = (u_1 \otimes u_2 \cdots \otimes u_{n-1}) \underbrace{(u_n \cdot v)}_{\text{scalar}}$$

Components of a tensor

order — 1 (vector) — 2 — m

$$v = v_i e_i \qquad T = T_{ij}\, e_i \otimes e_j \qquad T = T_{i_1 \ldots i_m}\, e_{i_1} \otimes \cdots \otimes e_{i_m} \quad (*)$$

$$v_i = v \cdot e_i \qquad T_{ij} = e_i \cdot (T e_j) \qquad T_{i_1 \ldots i_m} = e_{i_1} \cdot \Big[\underbrace{\underbrace{\big[\underbrace{(T e_{i_m})}_{m-1 \text{ order}}\, e_{i_{m-1}} \big] \cdots}_{m-2}\, e_{i_2} }_{\text{vector}} \Big]$$

See definition 46 for components of m'th order tensor (shown in red here)

Theorem 84 for equation (*)
Theorem 88 for coordinate transformation of mth order tensors

Theorem 88 for coordinate transformation of mth order tensors

Tensor product in general

$\overset{m}{T}$ mth order tensor

$\overset{n}{S}$ nth " "

(TS) $m-n$ order tensor

$$(TS)_{i_1 i_2 \ldots i_{m-n}} = T_{i_1 i_2 \ldots i_{m-n}\, i_{m-n+1} \ldots i_m} S_{i_m i_{m-1} i_{m-2} \ldots i_{m-n+1}}$$

We have identity matrices from m'th order to m'th order tensors

$m=1$ $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\overset{m}{T} = \overset{2m}{I}\,\overset{m}{T}$

$m=2$ $\overset{4}{I}_{ijkl} =$ δ's indices i, j, k, l