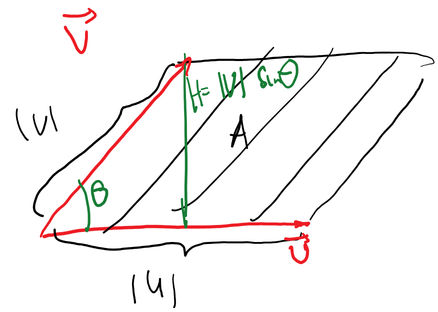
CM2021/09/21

Tuesday, September 21, 2021 4:27 PM

1.13 Vector cross (or exterior) product

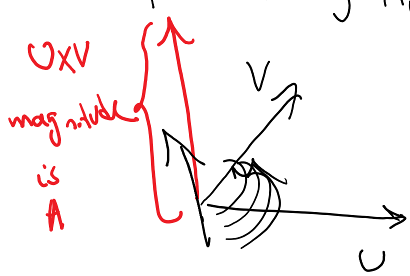
$\vec{u} \times \vec{v} = ?$



magnitude $|\vec{u} \times \vec{v}| = A$ area $A = |u| H = |u||v| \sin\theta$

direction is normal to u, v plane following right hand side rule:

recall $u \cdot v = |u||v| \cos\theta$



One can show that this definition is consistent with

$$\vec{u} \times \vec{v} = \underbrace{(\overset{3}{E} v)}_{\text{2nd order tensor}} u \rightarrow \text{vector}$$

$$\overset{3}{E} = \epsilon_{ijk}\, e_i \otimes e_j \otimes e_k$$

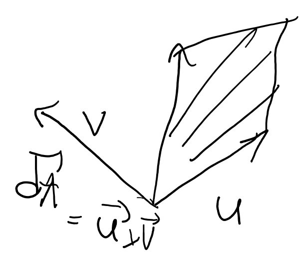
alternating tensor

Expression of $u \times v$ in a given coordinate system

$$\boxed{u \times v = \det \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \epsilon_{ijk}\, u_i\, v_j\, e_k}$$

Use of $u \times v$?

— Computing area



$v \times u = -u \times v$

$A_{tri} = \frac{|\vec{u} \times \vec{v}|}{2}$

$u \times (v \times w) \overset{?}{=} (u \times v) \times w$ No

It's not associative

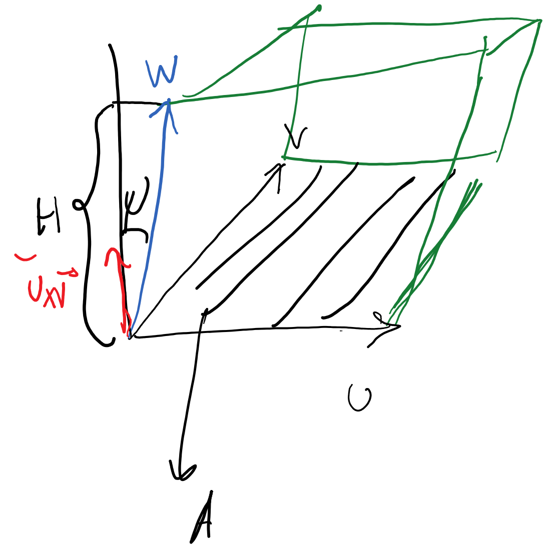
Theorem 93 *The vector product is* not associative*:*

$$(\mathbf{u}\times\mathbf{v})\times\mathbf{w} \neq \mathbf{u}\times(\mathbf{v}\times\mathbf{w}),$$

indeed

$$\begin{aligned}(\mathbf{u}\times\mathbf{v})\times\mathbf{w} &= (\mathbf{u}\cdot\mathbf{w})\,\mathbf{v}-(\mathbf{v}\cdot\mathbf{w})\,\mathbf{u},\\ \mathbf{u}\times(\mathbf{v}\times\mathbf{w}) &= (\mathbf{u}\cdot\mathbf{w})\,\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\,\mathbf{w}.\end{aligned}$$

Triple product



$\bar{V}$

volume $= AH = A|w|\cos\psi$

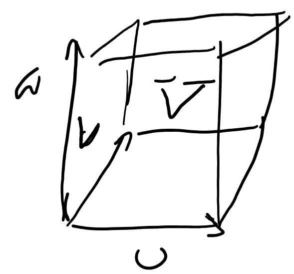
$= \underline{|u\times v||w|\cos\psi}$

ψ is angle between $u\times v$ & w

$(u\times v)\circ w$

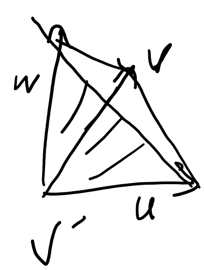
$$= \begin{vmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \cdot (w_1 e_1 + w_2 e_2 + w_3 e_3)$$

$$(u\times v)\circ w = \det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$



vol of ~~tetra~~ tetrahedron V'

$$= \frac{(u\times v).w}{6}$$



$(u\times v)\cdot w > 0$

they're called to be positively oriented (or follow the RH rule) they're called to be positively oriented (or follow the RH rule)

$(u \times v)\cdot w > 0$

+

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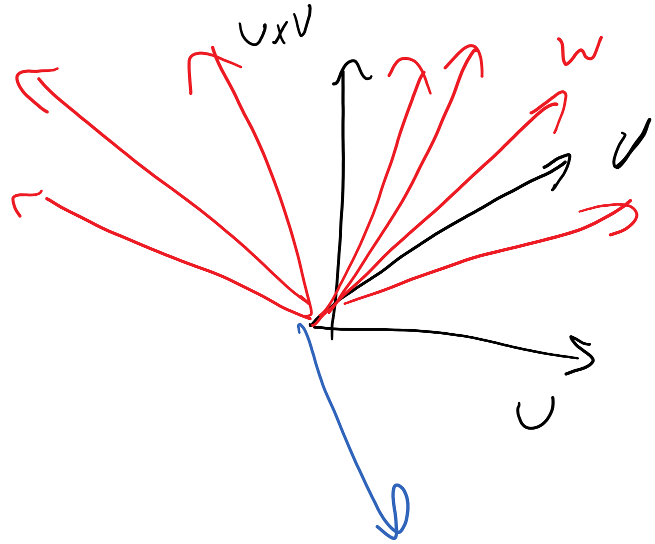
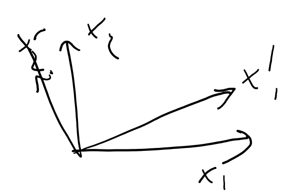


Special types of 2nd order tensors:
1. Orthonormal
2. Skew symmetric
3. Symmetric
4. Positive definite

Orthonormal recall for matrix Q we had $QQ^t = Q^tQ = I$



For 2nd order orthogonal tensors we have a similar relation:

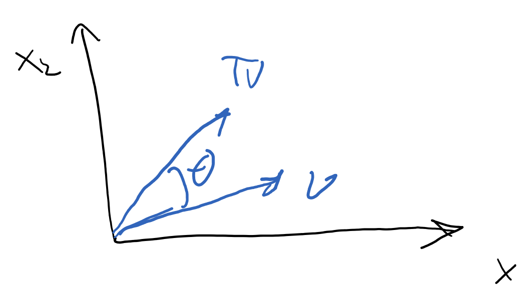
$$\boxed{\begin{gathered} TT^t = T^tT = I \\ T^t = T^{-1} \end{gathered}}$$

What does an orthonormal tensor represent?

rotation operation

$v \longrightarrow \underline{Tv}$

v rotated by θ around origin

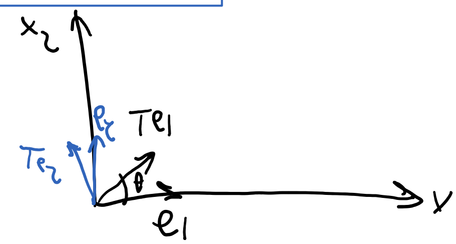


v rotated by θ around origin

Expression of T in (e_1, e_2) system

$$\underbrace{\begin{bmatrix} T_{11} \\ T_{21} \end{bmatrix}}^{C_1} \neq \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \overbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}^{e_1}$$

$$\boxed{Col_j(T) = T\, e_j}$$



$$Te_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = col_1(T)$$

$$Te_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = col_2(T)$$

$$\text{rotation } T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

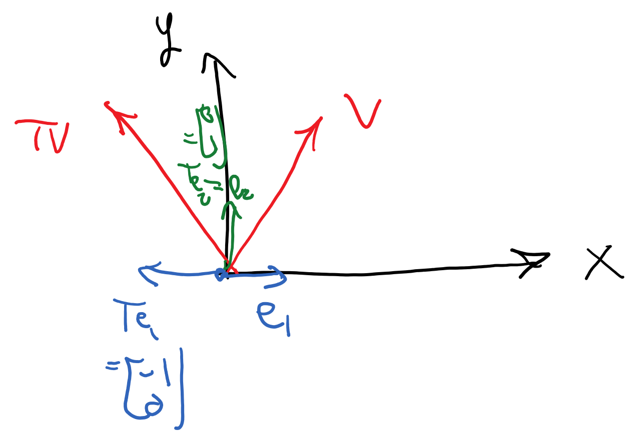
$\det T = 1$

Rotation is an orthogonal operation

Another example

Reflection

e.g w.r.t y axis



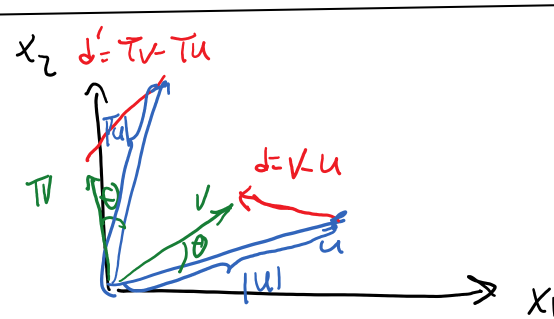
$$\text{reflection } T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\det T = -1$

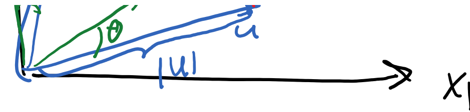
HW rotation and/or reflection orthogonal tensor

Orthogonal operators preserve
- Length — $|Tu| = |u|$
- Angle
- Distance $d' = d$ $|v - u| = |Tv - Tu|$

Of vectors



Theorem



Theorem

The following statements are equivalent

1. $T \in \text{Orth}\ V$
2. $\forall u, v \quad Tu \cdot Tv = u \cdot v$ preserve inner product
3. $\forall u \quad |Tu| = |u|$ " magnitude
4. $\forall u, v \quad |Tu - Tv| = |u - v|$ " distance

$b \cdot Ta = T^t b \cdot a$

$1 \Longleftrightarrow 2$ $\quad Tu \cdot Tv = \underbrace{T^t T}_{I} u \cdot v = u \cdot v$

1 $T \in \text{Orth}\ V \Rightarrow T^t T = I$

$2 \Rightarrow 3$ $\quad |Tu| = \sqrt{Tu \cdot Tu} \overset{\text{from 2}}{=} \sqrt{u \cdot u} = |u|$

$3 \Rightarrow 2$

$|Tu| = |u|$

$|Tv| = |v|$

$|T(u+v)| = |u+v|$

square all of these and rearrange them $\rightarrow Tu \cdot Tv = u \cdot v$

$3 \rightarrow 4$ $\quad \overset{3}{|Tu| = |u|}$ choose $u - v$ for u

$|T(u-v)| = |u-v|$

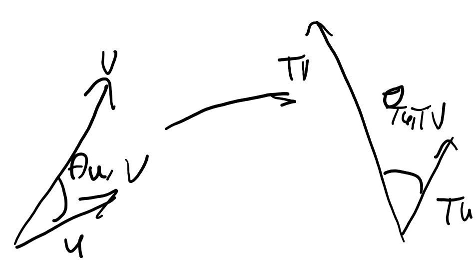
$4 \rightarrow 3$ $\quad |T(u-v)| = |u-v|$ choose $v = 0 \rightarrow |Tu| \overset{3}{=} |u|$

As I mentioned, orthogonal tensors preserve angle as well.
Why?

$$\theta_{Tu,Tv} = \theta_{u,v}$$

$$\cos\theta_{Tu,Tv} = \frac{Tu \cdot Tv}{|Tu||Tv|} \overset{\text{prob 2}}{=} \frac{u \cdot v}{|u||v|} = \cos\theta_{u,v}$$

prop 3



$\rightarrow$ angle is preserved.

Skew-symmetric tensors:
They'll represent "small rotations"

$$W^t = -W \qquad W_{ji} = -W_{ij} \qquad u_j u_i = u_i u_j \qquad \text{HW?}$$

$$u \cdot Wu = u_i (Wu)_i = u_i W_{ij} u_j = u_i u_j W_{ij} = 0$$

another way to show this

$$\underbrace{u \cdot Wu}_{A} = (\underbrace{W^t}_{=-W} u) \cdot u = ((-W)u) \cdot u = -Wu \cdot u = -u \cdot Wu$$

$$A = -A \rightarrow 2A = 0 \rightarrow A = 0 \qquad \boxed{u \cdot Wu = 0}$$

if W is skew symmetric

$$W_{ji} = -W_{ij}$$

$$W_{\underline{i}i} = -W_{\underline{ii}} \qquad \text{no summation on } i$$

$$\downarrow W_{\underline{ii}} = 0 \qquad \text{diagonals are zero}$$

in 3D we have

$$W = \begin{bmatrix} 0 & W_{12} & -W_{31} \\ -W_{12} & 0 & W_{23} \\ W_{31} & -W_{23} & 0 \end{bmatrix}$$

$$Wu = \begin{bmatrix} 0 & W_{12} & -W_{31} \\ -W_{12} & 0 & W_{23} \\ W_{31} & -W_{23} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} W_{12}u_2 - W_{31}u_3 \\ -W_{12}u_1 + W_{23}u_3 \\ W_{31}u_1 - W_{23}u_2 \end{bmatrix}$$

$$= \begin{bmatrix} \omega_{23} \\ \omega_{31} \\ \end{bmatrix} \times \begin{bmatrix} u_1 \\ u_2 \\ \end{bmatrix}$$

you can check it at home

$$= \begin{bmatrix} \omega_{31} \\ \omega_{12} \end{bmatrix} \times \begin{bmatrix} v_2 \\ v_3 \end{bmatrix} \; !$$

you can check it at home

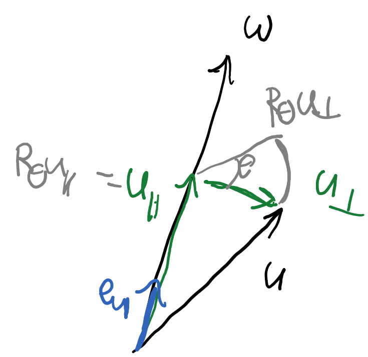
$$\boxed{\begin{array}{l} W u = w \times u \\ \text{1-1 correspondence between Skew tensors } W \\ \text{\& their corresponding "small rotation" vectors } w \\ w_i = -\frac{1}{2}\epsilon_{ijk} W_{jk} \\ W_{ij} = -\epsilon_{ijk} w_k \end{array}}$$

Why $\omega \times u$ can represent small rotations

$\omega = |\omega| e_\omega$

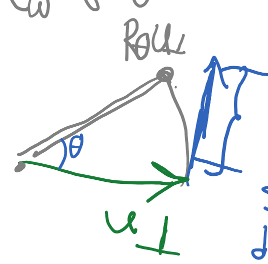
$u = u_\parallel + u_T$

$R_\theta u = \underbrace{R_\theta}_{\text{rotation with angle } \theta \text{ along } e_\omega} u_\parallel + R_\theta u_T$



look from the top.

e_ω going out of the plane



$Wu = \omega \times u$
small rotation difference