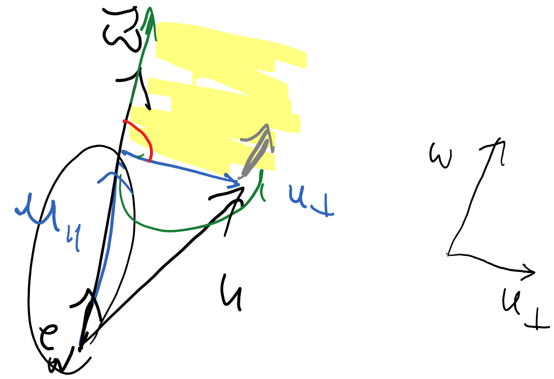
CM2021/09/23

Thursday, September 23, 2021 4:28 PM

Small angle rotation and relation to skew matrices

Recall $Wu = \omega \times u$

Look from the top



$$\omega \times u = \cancel{\omega \times u_{\parallel}}^{\,0} + \omega \times u_\perp \quad e_\omega$$

$$|\omega \times u_\perp|$$

$$= |(|\omega| e_\omega) \times u_\perp|$$

$$= \underbrace{|\omega|}_{\text{assume this is small} = \theta} \underbrace{|e_\omega|}_{1} \underbrace{|u_\perp|}_{\ell} \underbrace{\sin\theta_{e_\omega, u_\perp}}_{1 \;(90)} = \theta\ell$$

real rotation $\vec{y} = Q\vec{x}$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \ell \\ 0 \end{bmatrix}$$

$$\omega \times u \approx \underline{\vec{y} - \vec{x}} = \vec{\omega} \times \vec{u} = \begin{bmatrix} 0 \\ 0 \\ \theta \end{bmatrix} \times \begin{bmatrix} \ell \\ 0 \\ 0 \end{bmatrix}$$

Skew matrices and their corresponding vector w, represent the change of location for small angle rotation (when the magnitude of w $= \theta$ is small).

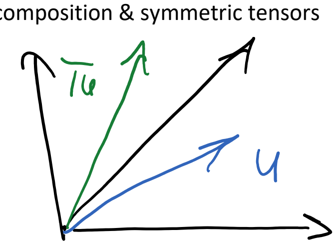
Recall that we can write any tensor as the summation of its symmetric and skew-symmetric parts:

$$T = S + W \qquad S = \frac{T + T^t}{2} \qquad W = \frac{T - T^t}{2}$$

How about symmetric tensors?

Eigen-decomposition & symmetric tensors

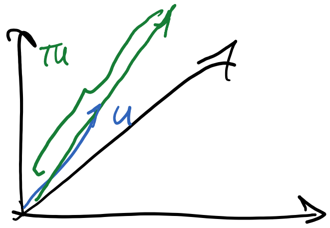
Eigenvectors & values



What about the case that $Tu \parallel u$

$Tu \parallel u$

$$\underset{\text{eigenvalue}}{\lambda}\ \ Tu = \lambda u \quad \underset{\text{eigenvector}}{u}$$



for an $n \times n$ matrix we have a maximum of n linearly independent (eigenvector, value) pairs

$$T u_{(1)} = \lambda_1 u_{(1)}$$

$$T u_{(2)} = \lambda_2 u_{(2)}$$

$$\vdots$$

$$T u_{(i)} = \lambda_i u_{(i)}$$

no summation on i

$$T \underbrace{\begin{bmatrix} u_{(1)} & u_{(2)} & \cdots & u_{(n)} \end{bmatrix}}_{U} = \begin{bmatrix} \lambda_1 u_{(1)} & \lambda_2 u_{(2)} & \cdots & \lambda_n u_{(n)} \end{bmatrix}$$

$$= \begin{bmatrix} u_{(1)} & \cdots & u_{(n)} \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}}_{\Lambda}$$

$$TU = U\Lambda$$

$$T = U \Lambda U^{-1} \qquad \text{diagonalized form of } T$$

$$T^2 = (U\Lambda U^{-1})(U\Lambda U^{-1}) = U\Lambda(U^{-1}U)\Lambda U^{-1} = U\Lambda I \Lambda U^{-1} = U\Lambda^2 U^{-1}$$

Similarly $\quad T^n = U \Lambda^n U^{-1}$

e.g. $n = 100$

Recall $e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{3!} + \cdots$

$$e^{T} = I + T + \frac{1}{2!}T^2 + \frac{1}{3!}T^3 + \cdots =$$

$$U\Lambda^0 U + U(\Lambda)U^{-1} + U\left(\frac{1}{2}\Lambda^2\right)U^{-1} + \cdots$$

$$= U \begin{bmatrix} 1+\lambda_1+\frac{\lambda_1^2}{2}+\cdots & & \\ & 1+\lambda_2+\lambda_2^2/2+\cdots & \\ & & 1+\lambda_n+\frac{\lambda_n^2}{2}+\cdots \end{bmatrix} U^{-1}$$

$$e^{\lambda} = U \begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix} U^{-1}$$

$$\boxed{f(T) = U \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix} U^{-1}}$$

Are all square matrices diagonalizable?

The matrix must have n linearly independent eigenvectors.

If a matrix has n distinct eigenvalues -> the corresponding eigenvectors are linearly independent

$T_{n\times n}$ $\quad \lambda_1 \neq \lambda_2 \neq \lambda_3 \cdots \neq \lambda_n$

$\lambda_i \neq \lambda_j$ for $i \neq j$

The problem can arise if the matrix has repeated eigenvalues

$$T = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

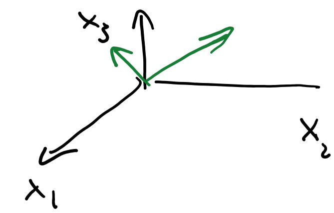
algebraic multiplicity

$\lambda_{1,2} = 3 \rightarrow$ 2 eigenvalues 3   eigenvectors $\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$ geometric multiplicity

$\lambda_3 = 5 \rightarrow$

eigenvector $= \begin{bmatrix} 1 \\ \cdot \\ \cdot \end{bmatrix}$ $x_3 \uparrow$

eigen vector $\begin{bmatrix}1\\0\\0\end{bmatrix}$



Example of non diagonalizable matrix

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$\lambda_1 = \lambda_2 = 1$ eigenv. 1 has Algebraic multiplicity of 2

$m_A(1) = 2$

Finding eigen vector

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \overset{\lambda}{1} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

the only valid $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ slu is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Geometric multiplicity of T for $\lambda = 1$ is 1 $\quad m_G(1) = 1$

This matrix is not diagonalizable

Further reading

Jordan Form

$$\begin{bmatrix} \lambda_1 & & & & & & & & \\ & \lambda_2 & & & & & & & \\ & & \lambda_3 & & & & & & \\ & & & \ddots & & & & & \\ & & & & \lambda_m & 1 & & & \\ & & & & & \lambda_m & 1 & & \\ & & & & & & \lambda_m & 1 & \\ & & & & & & & \lambda_m & \\ & & & & & & & & \lambda_m \end{bmatrix}$$

Done with the background, we focus on symmetric matrices:

Example in 3D

$$\begin{bmatrix} a & b & c \end{bmatrix}$$

Example in 3D

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

$A^T = A$

Some properties of symmetric matrices:

1. Eigenvalues are real:

$$\underbrace{\overbrace{\bar{u} \cdot Au}^{\text{Real}}}_{\text{conjugate}} = \underbrace{\lambda}_{\in \mathbb{R}} \overbrace{\bar{u}u}^{>0}$$

$Au = \lambda u$ — $\lambda \in \mathbb{C}$ (complex) in general

for A symmetric $\lambda \in \mathbb{R}$

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2. For distinct eigenvalues, their corresponding eigenvectors are normal to each other

a) $A u_{(1)} = \lambda_1 u_{(1)}$

b) $A u_{(2)} = \lambda_2 u_{(2)}$

$A = A^T$

$\left.\lambda_1 \neq \lambda_2\right\} \rightarrow u_{(1)} \perp u_{(2)}$

$\cdot$ of a) with $u_{(2)}$ $\quad u_{(2)} \cdot A u_{(1)} = \lambda_1 u_{(2)} \cdot u_{(1)} = \lambda_1 u_{(1)} \cdot u_{(2)}$

$\cdot$ of b) $= u_{(1)}$ $\quad u_{(1)} \cdot A u_{(2)} = \lambda_2 u_{(1)} \cdot u_{(2)}$

$\underbrace{A^t}_{A = A^t} u_{(1)} \cdot u_2 = A u_{(1)} \cdot u_{(2)} = u_{(2)} \cdot A u_{(1)} = \lambda_2 u_{(1)} \cdot u_{(2)}$

Subtract

$$\boxed{(\lambda_2 - \lambda_1)\, u_{(1)} \cdot u_{(2)} = 0 \quad \lambda_1 \neq \lambda_2}$$

$u_{(1)} \cdot u_{(2)} = 0 \qquad u_{(1)} \perp u_{(2)}$

Assume A n x n, symmetric, has n distinct eigenvalues -> then we can form n orthonormal eigenvectors:

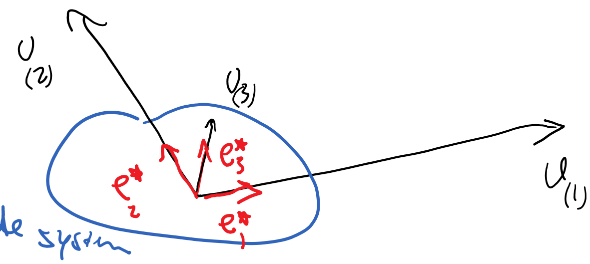
$A u_{(1)} = \lambda_1 u_{(1)} \quad \xrightarrow{\times \frac{1}{\|u_{(1)}\|}} \quad A \underbrace{\hat{e}_1}_{\text{unit vector for } u_{(1)}} = \lambda_1 \hat{e}_1$

Same with $u_{(2)}, \dots$

$\rightarrow \hat{e}_i$'s are eigenvectors of A

&

$$\boxed{e^*_i \cdot e^*_j = \delta_{ij}}$$



$A \rightarrow$ orthonormal coordinate system

I proved this when eigenvalues of symmetric A are distinct but this also holds when they have repeated eigenvalues

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2D

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{\text{if sym.}} A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} a-\lambda & b \\ b & d-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow$$

$$\det \begin{bmatrix} a-\lambda & b \\ b & d-\lambda \end{bmatrix} = 0 \rightarrow \lambda^2 - (a+d)\lambda + ad - b^2 = 0$$

$$\lambda = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-b^2)}}{2}$$

$$\lambda_{1,2} = \frac{a+d \pm \sqrt{(a-d)^2 + 4b^2}}{2}$$

When do we have repeated eigenvalues

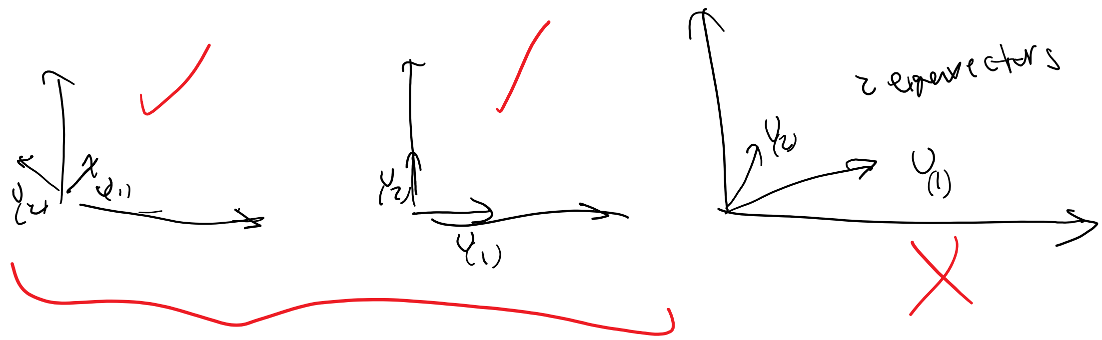
$(a-d)^2 + 4b^2 = 0 \longrightarrow \boxed{\begin{matrix} a=d \\ b=0 \end{matrix}}$

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \longrightarrow \lambda_{1,2} = a$$

$$\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{A-\lambda I} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad !$$

$\longrightarrow$ any $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is an eigenvector

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \qquad Au = (aI)u = au$$



we can choose any of these as orthonormal eigenvectors

3D

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

$\det(A - \lambda I) = 0$

3rd order eqn for λ

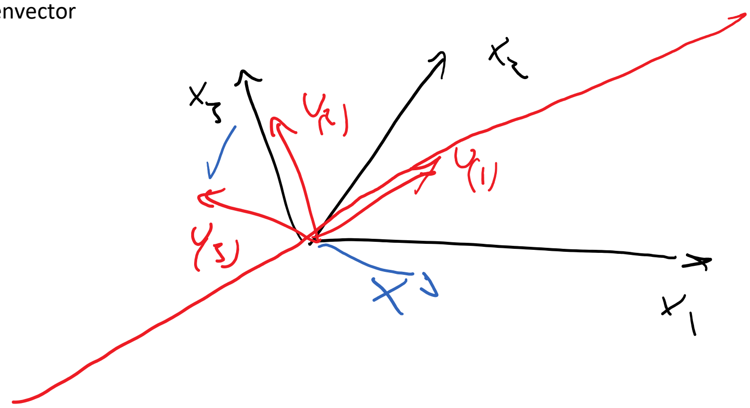
$\lambda_1, \lambda_2, \lambda_3$

Case 1 : $\lambda_1 \neq \lambda_2 \quad \lambda_2 \neq \lambda_3 \quad \lambda_1 \neq \lambda_3$

Since each eigenvalue is distinct -> it has a distinct eivenvector (sized to 1):

x_3

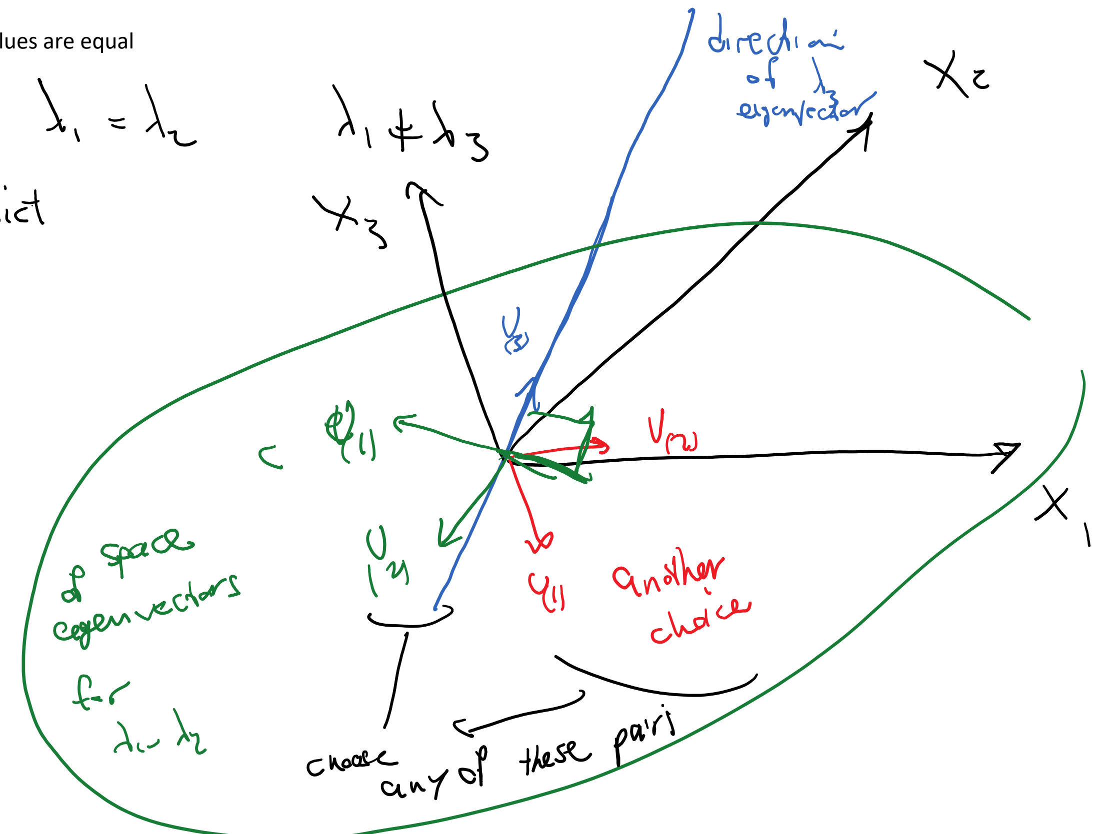
Since each eigenvalue is distinct -> it has a distinct eivenvector (sized to 1):



Case 2: two eigenvalues are equal

$\lambda_1 = \lambda_2 \qquad \lambda_1 \neq \lambda_3$

λ_3 is distict



$v_{(i)} \perp v_{(j)} \qquad i \neq j \qquad v_{(i)} \cdot v_{(j)} = \delta_{ij}$

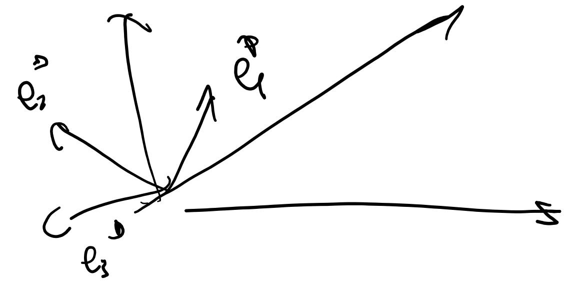
Case 3

$\lambda_1 = \lambda_2 = \lambda_3 = a$

$T = a\,I$ any vector is an eigen vector

Just 3 arbitrary mutually orthonormal vectors

Just 3 arbitrary ...



For an nxn symmetric matrix:

1. It is always diagonalizable
2. Eigenvalues are real
3. We can always find n orthonormal eigenvectors.

How to calculate eigenvectors of a symmetric 3 x 3 matrix:

$$(S - \underbrace{\sigma}_{\text{eigenvalue}} I)u = 0$$

$$\det \begin{bmatrix} S_{11} - \sigma & S_{12} & S_{13} \\ S_{21} = S_{12} & S_{22} - \sigma & S_{23} \\ S_{31} = S_{13} & S_{32} = S_{23} & S_{33} - \sigma \end{bmatrix} = 0$$

Cayley-Hamilton equation:
Characteristic equation of the tensor:

$$-\sigma^3 + I_1\,\sigma^2 - I_2\,\sigma + I_3 = 0$$

$$I_1 = tr(S) = S_{11} + S_{22} + S_{33}$$

$$I_2 = \frac{1}{2}\left[(tr(S))^2 - tr(S^2)\right] = \frac{1}{2}\left(S_{ii}S_{jj} - S_{ij}S_{ij}\right)$$

$$I_3 = \det |S_{ij}|$$