CM2021/09/28

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Cayley-Hamilton equation:
Characteristic equation of the tensor:

$$-\sigma^3 + I_1 \sigma^2 - I_2 \sigma + I_3 = 0$$

$$I_1 = tr(S) = S_{11} + S_{22} + S_{33}$$

$$I_2 = \frac{1}{2}\left[(tr(S))^2 - tr(S^2)\right] = \frac{1}{2}\left(S_{ii} S_{jj} - S_{ij} S_{ij}\right)$$

$$I_3 = \det |S_{ij}|$$

by noting that $S = U \sigma U^{-1}$ the diagonalized form

$$-S^3 + I_1 S^2 - I_2 S + I_3 I = U \begin{bmatrix} \underbrace{-\sigma_1^3 + I_1 \sigma_1^2 - I_2 \sigma_1 + I_3}_{=0} & 0 & 0 \\ 0 & \underbrace{-\sigma_2^3 \cdots}_{=0} & 0 \\ & & \underbrace{-\sigma_3^3 + I_1 \sigma_3^2 \cdots}_{=0} \end{bmatrix} U^{-1}$$

$$= 0_{3\times 3}$$

Cayley-Hamilton theorem:
For a diagonizable matrix, it too satisfies its characteristic equation. For these symmetric 3x3 matrices

$$-\sigma^3 + I_1 \sigma^2 - I_2 \sigma + I_3 1 = 0$$

$$\downarrow$$

$$-S^3 + I_1 S^2 - I_2 S + I_3 \underset{\substack{\downarrow \\ S^0}}{I} = 0$$

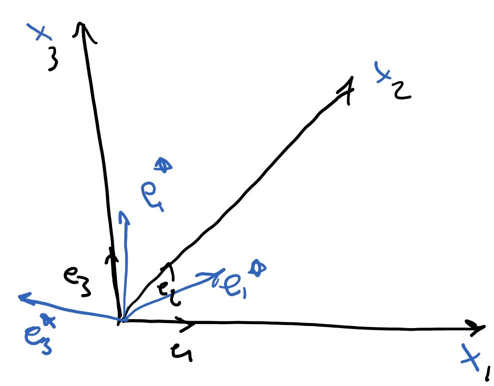
Representation of symmetric tensors

$$e_1, e_2, e_3 \to S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix}$$

(1) $S e_i^* = \underset{\substack{\downarrow \\ \text{no summation on } i}}{\sigma_i} e_i^*$

$\sigma_i \in \mathbb{R}$

$e_i^* \cdot e_j^* = \delta_{ij}$



(e') is another orthonormal coordinate system $\longrightarrow$

we can do coordinate transformation

Define $\quad Q = \begin{bmatrix} e^*_1 \\ \hline e^*_2 \\ \hline e^*_3 \end{bmatrix}$ expressed in (e_1, e_2, e_3)

$$T^*_{ij} = Q_{im} Q_{jn} T_{mn}$$

Similarly $\quad S^*_{ij} = Q_{im} Q_{jn} S_{mn}$

Question: What are the components of S in e^*_1, e^*_2, e^*_3 system

Recall $\quad T_{ij} = e_i \cdot T e_j \quad$ components of T in e_1, e_2, e_3 system

Similarly $\quad S^*_{ij} = e^*_i \cdot \underbrace{(S e^*_j)}_{\substack{= \sigma_j e^*_j \\ \text{eqn 1}}} = \underbrace{e^*_i \cdot (\sigma_j e^*_j)}_{} =$

$\sigma_j e^*_j$ no summation

$\sigma_j \underbrace{(e^*_i \cdot e^*_j)}_{\delta_{ij}}$

$= \delta_{ij} \sigma_j$ no summation on j

$S^*_{11} = \delta_{11} \sigma_1 = \sigma_1$

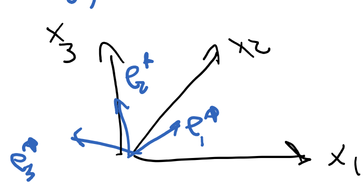
$$[S]^* = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

$[S]^* = \text{diagonal}(\sigma_1, \sigma_2, \sigma_3)$

$S = S^*_{ij} e^*_i \otimes e^*_j$

$= \sum_{i=1}^{3} \sigma_i e^*_i \otimes e^*_i$

$= \sigma_i e^*_i \otimes e^*_i$



Numerical example:

The figure shows the eigenvalues and vectors of S,

Provide the components of S in e1, e2 and e*1, e*2 systems

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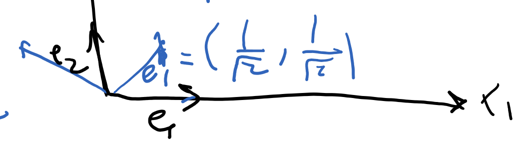
Provide the components of S in e1, e2 and e*1, e*2 systems

$\sigma_2 = 10$ $\qquad$ $\sigma_1 = 5$

$\underline{\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = e_2^*}$ $\qquad$ $e_1^* = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$



$$[S]^*_{e_2^* e_1^*} = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} = \sigma_1 e_1^* \otimes e_1^* + \sigma_2 e_2^* \otimes e_2^*$$

expression of this in $e_1 e_2$ system

$$[S]_{e_1 e_2} = \begin{bmatrix} & \\ & \end{bmatrix} = 5 \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \otimes \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$+10 \left(-\frac{1}{\sqrt{2}}, +\frac{1}{\sqrt{2}}\right) \otimes \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$= 5 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} + 10 \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ +\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

recall

$$U \cdot V = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$U \otimes V = \begin{bmatrix} U_1 \\ V_2 \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}$$

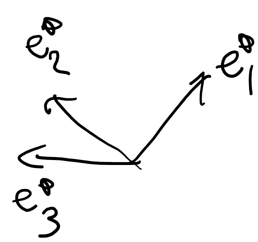
$$= 5 \begin{bmatrix} 1/2 & \frac{1}{2} \\ 1/2 & 1/2 \end{bmatrix} + 10 \begin{bmatrix} +\frac{1}{2} & -1/2 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} =$$

$$\begin{bmatrix} 15/2 & -\frac{5}{2} \\ -5/2 & \frac{15}{2} \end{bmatrix}$$

Second example:

$$S = \sum_i \sigma_i e_i^* \otimes e_i^*$$

$$[S]^* = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix}$$



$\det S = \sigma_1 \sigma_2 \sigma_3$

$\mathrm{trace}\, S = \sigma_1 + \sigma_2 + \sigma_3$

$$S^{-1} = \sum \frac{1}{\sigma_i} e_i^* \otimes e_i^*$$

$\underline{\text{if } \det S \neq 0}$

$$[S^{-1}]^* = \begin{bmatrix} 1/\sigma_1 & & \\ & 1/\sigma_2 & \\ & & 1/\sigma_3 \end{bmatrix}$$

verify $[S]^*[S^{-1}]^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\ln S = \sum$ ...

$$\ln \mathbf{S} = \sum_i \ln \sigma_i \, e_i \otimes e_i \qquad [\ln \mathbf{S}]^{*} = \begin{bmatrix} \ln\sigma_1 & 0 & 0 \\ & \ln\sigma_2 & \\ & & \ln\sigma_3 \end{bmatrix}$$

Positive definite tensor:

$$u \cdot T u = u \cdot (\underbrace{\mathrm{Sym}\,T}_{\frac{T+T^t}{2}} + \underbrace{\mathrm{Skew}\,T}_{\frac{T-T^t}{2}})\, u = u \cdot (\mathrm{Sym}\,T) u + \underbrace{u \cdot (\mathrm{Skew}\,T) u}_{0}$$

$$\underbrace{u_i \, (\mathrm{Skew}\,T)_{ij} \, u_j}_{\mathrm{Sym}}$$

$$\boxed{u \cdot T u = u \, \mathrm{Sym}\,T \cdot u}$$

Skew-symmetric part of the tensor T does not contribute to u.Tu

Many times because of this, we directly work with symmetric tensors in forming such products

A positive definite tensor T satisfies the following properties:

$$\forall u \quad u \cdot T u \geq 0 \quad \& \quad u \cdot T u = 0 \Leftrightarrow u = 0$$

A positive or semi-positive tensor only satisfies the first condition

$$\forall u \quad u \cdot T u \geq 0$$

Examples

$$T = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$$

not positive definite
but it's positive

$$[0 \ \ 1] \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \quad \text{but} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq 0$$

$$T = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \qquad u \cdot T u = [u_1 \ \ u_2] \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} =$$

$$\overset{>0}{5} \underbrace{u_1^2}_{\geq 0} + \overset{>0}{3} \underbrace{u_2^2}_{\geq 0} \geq 0$$

$$u \cdot Tu = 0 \equiv 5u_1^2 + 3u_2^2 = 0 \equiv \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

positive definite

$$T = \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix} \qquad u \cdot Tu = 5u_1^2 - 3u_2^2$$

choose $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\qquad u \cdot Tu = -3 < 0$

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remark 1

if T is diagonalizable & symmetric

$$T = U \Lambda U^t, \qquad \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$v \cdot Tv = v \cdot (U \Lambda U^t) v$$

$$= \underbrace{U^t v}_{w} \cdot \Lambda \underbrace{(U^t v)}_{w}$$

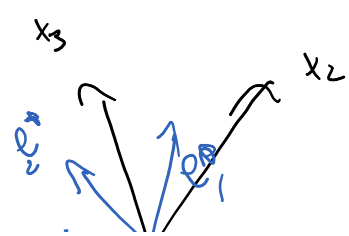
$$w \cdot \Lambda w = [w_1 \cdots w_n] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

$$= w_1^2 \lambda_1 + w_2^2 \lambda_2 + \cdots + w_n^2 \lambda_n$$

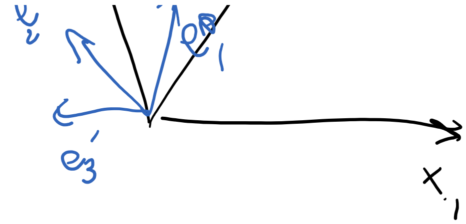
$$\geq 0 \quad \text{iff} \quad \lambda_1, \ldots, \lambda_n \geq 0$$

For a symmetric tensor positive definiteness $\equiv$ eigenvalues > 0

$$[S]^* = \begin{bmatrix} s_1 & & 0 \\ & \ddots & \end{bmatrix}$$



$$[S]^{T} = \begin{bmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_n \end{bmatrix}$$



Positive definiteness , $s_i > 0$

Positive $s_i \geq 0$

Side note:

We can define a norm once we have a positive definite matric $\|u\|_T = \sqrt{u \cdot T u}$

FYI used in structural dynamics $M\ddot{U} + K\dot{U} = F$

$M > 0$ mode shapes ϕ

$\|\phi\|_M = \sqrt{\phi \cdot M \phi}$

Question:

Is there any easy way to form a symmetric & positive (or positive def) tensor?

$$\boxed{\underbrace{F}_{\text{general 2nd order tensor}} \longrightarrow \underbrace{C}_{\text{symmetric}} = F^t F \begin{matrix} > 0 \text{ if } \det F \neq 0 \\ \geq 0 \end{matrix}}$$

$C^t = (F^t F)^t = F^t (F^t)^t = F^t F = C$ so C is symmetric

positive check $\sqrt{a \cdot a \cdot Tb = 1}$

positive check

$$\boxed{a \cdot Tb = T^t a \cdot b}$$

$$v \cdot Cu = v \cdot F^t F u = (F^t)^t u \cdot Fu =$$

$$\underbrace{Fu}_{v} \cdot \underbrace{Fu}_{v} \geq 0$$

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checking the definite part

$$u \cdot Cu = 0 \qquad Fu \cdot Fu = 0 \longrightarrow Fu = 0 \longrightarrow u = 0$$

($\det F \neq 0$

$F^{-1} F u = 0 \to u = 0$)

else there are some $v \neq 0$ for which $Fv = 0$

$$\sqrt{9} = \pm 3$$

$9 > 0$ I just chose the positive root

$$\sqrt{9} = 3$$

$S \geq 0$ can we define a real square root

$$S = \sigma_1 e_1^{*} \otimes e_1^{*} + \sigma_2 e_2^{*} \otimes e_2^{*} + \sigma_3 e_3^{*} \otimes e_3^{*} \qquad [S]^{*} = \begin{bmatrix} \sigma_1 \geq 0 & 0 & 0 \\ 0 & \sigma_2 \geq 0 & 0 \\ 0 & 0 & \sigma_3 \geq 0 \end{bmatrix}$$

$$U = \sqrt{S} \quad (\quad U^2 = S)$$

$$[U]^{*} = \begin{bmatrix} +\sqrt{\sigma_1} & 0 & 0 \\ 0 & \sqrt{\sigma_2} & 0 \\ 0 & 0 & \sqrt{\sigma_3} \end{bmatrix}$$

U is the unique Pdef square root of S

Theorem 112 (Polar Decomposition Theorem) *Let* $\mathbf{F} \in \text{Inv}\,\mathcal{V}$. *Then* $\exists$ *a unique pair of tensors* $\mathbf{U}, \mathbf{V} \in \text{Psym}$ *and a unique* $\mathbf{R} \in \text{Orth}\,\mathcal{V} \ni$

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}.$$

Moreover, $\det \mathbf{R} = +1$ *or* $\det \mathbf{R} = -1$, *depending as* $\det \mathbf{F} > 0$ *or* < 0.

Motivation for the proof

if

$$\boxed{F = RU} \rightarrow F^t F = (RU)^t RU = (U^t R^t) RU$$

$$U^t \underbrace{(R^t R)}_{\substack{I \\ R \in \text{Orth}\,\mathcal{V}}} U = \underbrace{U^t U}_{U \text{ is sym}} = U^2$$

$$F^t F = U^2$$

this gives the motivation on how to define U

$C = F^t F$ this is sym positive

then we can define its sym positive root

$$U := \sqrt{C} = \sqrt{F^t F} \equiv U^2 = F^t F$$

already proved to be sym & positive

we want to have

we defined this as $U = \sqrt{F^t F}$

$$F = RU$$

we are given this tensor

define $\boxed{R = F U^{-1}}$ obviously $F = RU$ ✓ $U \in Psym$ ✓

prove that R is orthogonal $R^t R = I$

$$R^t R = (F U^{-1})^t (F U^{-1}) = (U^{-t} F^t)(F U^{-1}) =$$

$$\underbrace{(U^t)^{-1}}_{U^{-1}} \underbrace{(F^t F)}_{U^2} U^{-1} = U^{-1} U^2 U^{-1} = I$$

$U = U^t$

R is orthogonal

$$\boxed{\begin{array}{l} F: \quad C = F^t F \to U = \sqrt{F^t F} \\ \qquad R = F U^{-1} \qquad R^t R = I \\ F = R U \quad R \to \text{orth}, \ U \to Psym \end{array}}$$

second part $\quad F = RU = VR$

if this holds

$$V = RUR^{-1} = \boxed{RUR^t}$$

R is orthogonal $\quad R^t = R^{-1}$

is V pos sym.

$$x \cdot Vx = x \cdot RUR^t x = \underbrace{R^t x}_{y} \cdot U\underbrace{(R^t x)}_{y} = y \cdot Uy \geq 0$$

U is pos

V is pos

$$V^t = (RUR^t)^t = (R^t)^t U^t R^t = RUR^t = V$$

U is sym

$F = RU = VR$

sym & positive

$$V = RUR^t \rightarrow V^2 = (RUR^t)(RUR^t)$$

$$= RU\underbrace{(R^t R)}_{I}UR^t$$

$$= R\underbrace{U^2}_{F^t F}R^t = RF^t F R^t$$

recall

$F = RU \rightarrow R = FU^{-1}$

$$\rightarrow V^2 = FU^{-1}\underbrace{(F^t F)}_{U^2}\underbrace{U^{-t}}_{U^{-1}}F^t$$

$$= FU^{-1}U^2U^{-1}F^t = FF^t$$

Rotation and/or reflection

Rotation and/or reflection

$$\boxed{\begin{aligned} F &= RU = VR \quad \text{stretches} \\ U &= \sqrt{C} \qquad C = F^t F \\ V &= \sqrt{B} \qquad B = F F^t \end{aligned}} \geq 0$$