CM2021/10/05

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e.g. stress or strain tensor

Theorem 106 *Let*

$$f(S_{ij}) := f(S_{11}, S_{22}, S_{33}, S_{12}, S_{13}, S_{23})$$

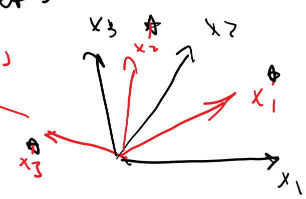
be a scalar invariant of $\mathbf{S} \in \mathrm{Sym}$ *(that is* $f(S_{ij}) = f(S'_{ij})$, *where* S_{ij} *and* S'_{ij} *are components of* $\mathbf{S}$ *w.r.t. two frames* X *and* X'*). Then* $\exists$ *a unique real-valued function* g *of three real variables* $\ni$

$$f(S_{ij}) = g(I_1(\mathbf{S}), I_2(\mathbf{S}), I_3(\mathbf{S}))$$

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{pmatrix}$$

$S \in \mathrm{Sym}$ → it can be expressed in principal directions

orthonormal eigen directions



$$[S^*] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

$f(S) = f(\sigma_1, \sigma_2, \sigma_3, 0, 0, 0)$ — f does not depend on coordinate

$= f_\sigma(\sigma_1, \sigma_2, \sigma_3)$ — "e.g. isotropic material"

we can more concisely express the relation in terms of principal values

$$\text{✱} \quad -\sigma^3 + I_1\sigma^2 - I_2\sigma + I_3 = 0$$

$$\sigma_1, \sigma_2, \sigma_3 \Longleftrightarrow I_1, I_2, I_3$$

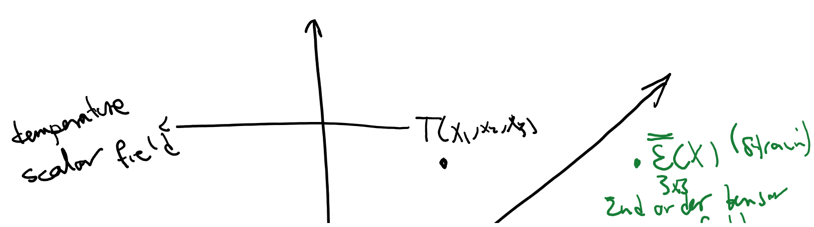
$I_1 = \sigma_1 + \sigma_2 + \sigma_3 = \mathrm{trace}\, S$ $\qquad$ $I_3 = \sigma_1\sigma_2\sigma_3 = \det S$

it's easier to use tensor invariants I_1, I_2, I_3 rather than $\sigma_1, \sigma_2, \sigma_3$ (we don't need to solve third order eqn ✱)

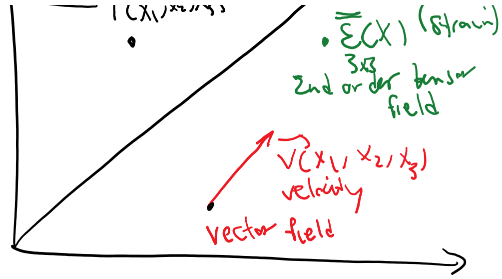
$$f(S) = f_I(I_1, I_2, I_3)$$

calibrate model based on I_1, I_2, I_3 (e.g. plasticity)

Tensor fields: → dependence on coordinate



temp. scalar field



These are tensor functions that depend on space (or space & time) coordinates.

They are in all parts of continuum mechanics: kinematics, balance laws, etc.

$$\int_{\partial\Omega} \sigma \cdot n \, ds = \int_{\Omega} \rho b \, dv$$

div ↓

$$\int_{\Omega} \nabla \cdot \sigma \, dv = \int_{\Omega} \rho b \, dv$$

need to calculate divergence (& grad, curl, ...) of tensor fields

$$q = -K \nabla T$$

Partial derivatives

T is an m th order tensor

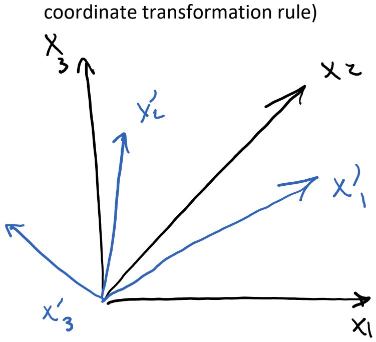
$$T = T_{i_1 \ldots i_m} \, e_{i_1} \otimes \cdots \otimes e_{i_m}$$

$$T_{i_1 \ldots i_m, j} = \frac{\partial T_{i_1 \ldots i_m}(x_1, x_2, x_3)}{\partial x_j}$$

$$\boxed{\nabla T = T_{i_1 \ldots i_m, j} \, e_{i_1} \otimes e_{i_2} \cdots \otimes e_{i_m} \otimes e_j}$$

Grad is always one tensor order higher

Since the definition is based on coordinate components, we need to show that it's coordinate-independent (i.e. it follows tensor coordinate transformation rule)



let's assume T is a 2nd order tensor

$$(\nabla T)_{ijk} = T_{ij,k} = \frac{\partial T_{ij}}{\partial x_k}$$

$$(\nabla T)'_{mnp} = \frac{\partial T'_{mn}(x_1', x_2', x_3')}{\partial x_p'}$$

what relation should we have between ∇T_{ijk} & $\nabla T'_{mnp}$ to verify it's a tensor

$\nabla T'$ Q Q Q ∇T .

check ∇T_{ijk} & $\nabla T'_{mnp}$

$$\nabla T'_{mnp} = Q_{mi} Q_{nj} Q_{pk} \nabla T_{ijk}$$

let's check this

chain rule

$$\nabla T'_{mnp} = \frac{\partial T'_{mn}}{\partial x'_p} = \frac{\partial\big(\overbrace{Q_{mi} Q_{nj} T_{ij}}^{T'_{mn}}\big)}{\partial x'_p} = Q_{mi} Q_{nj} \frac{\partial T_{ij}}{\partial x_k} \frac{\partial x_k}{\partial x'_p}$$

$x_k = \cancel{Q_{pk} x'_p}$

$x_k = Q_{rk} x'_r$ ✓

$$\nabla T'_{mnp} = Q_{mi} Q_{nj} \underbrace{\frac{\partial T_{ij}}{\partial x_k}}_{\nabla T_{ijk}} \frac{\partial Q_{rk} x'_r}{\partial x'_p}$$

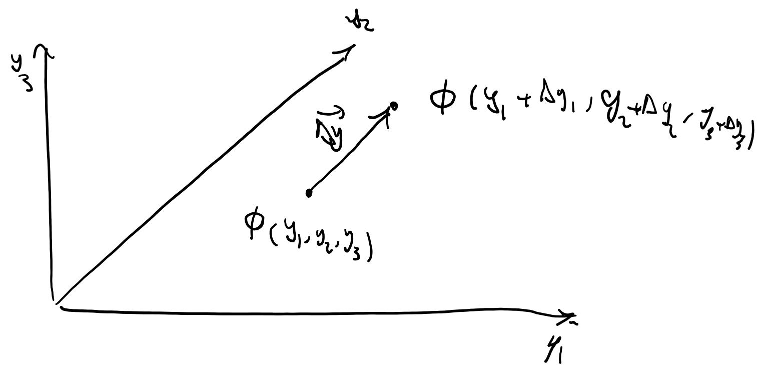
$$\nabla T'_{mnp} = Q_{mi} Q_{nj} Q_{rk} \nabla T_{ijk} \underbrace{\frac{\partial x'_r}{\partial x'_p}}_{\delta_{rp}}$$

$$\boxed{\nabla T'_{mnp} = Q_{mi} Q_{nj} Q_{pk} \nabla T_{ijk}}$$

So, gradient follows coordinate transformation rule -> it's a tensor (always one order higher than T)

Interpretation of gradient:

y is used for Global coordinate system



We know the change of location

$\vec{\Delta y}$

And we seek, change in the function

$$\Delta\phi = \phi(y+\Delta y) - \phi(y)$$

$$= \frac{\partial\phi}{\partial y_1}\Delta y_1 + \frac{\partial\phi}{\partial y_2}\Delta y_2 + \frac{\partial\phi}{\partial y_3}\Delta y_3 + \text{HOT}$$

$$= \underbrace{\nabla\phi}_{= [\partial\phi/\partial y_1 \;\; \partial\phi/\partial y_2 \;\; \partial\phi/\partial y_3]} \Delta y \to [\Delta y_1] \quad + \frac{1}{2}[\Delta y_1, \Delta y_2, \Delta y_3] \underbrace{\begin{bmatrix} \phi_{,11} & \phi_{,12} & \phi_{,13} \\ \phi_{,12} & & \\ & & \phi_{,33} \end{bmatrix}}\begin{bmatrix} \Delta y_1 \\ \Delta y_2 \\ \Delta y_3 \end{bmatrix} + \dots$$

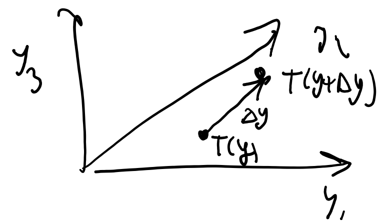
$$= \left[\partial\phi/\partial y_1 \quad \frac{\partial\phi}{\partial y_2} \quad \frac{\partial\phi}{\partial y_3}\right] \begin{bmatrix}\Delta y_1\\ \Delta y_2\\ \Delta y_3\end{bmatrix}$$

$$\underbrace{\left(\quad \phi_{,33} \right)}_{\text{Hessian matrix}} (\Delta y_3)^2 + \dots$$

$$\boxed{\Delta\phi = \nabla\phi\,\Delta y + \underbrace{O(\Delta y^2)}_{\text{terms of order } \Delta y^2 \text{ \& higher}} \approx \nabla\phi\,\Delta y \text{ as } \Delta y \to 0}$$

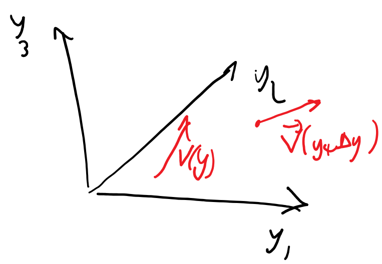
for any tensor T

$$\underbrace{\underbrace{\Delta T}_{\text{change in } T}}_{\text{order } m} \approx \underbrace{\nabla T}_{m+1 \text{ tensor}} \; \underbrace{\Delta y}_{\text{vector}}$$



$\Delta T = T(y+\Delta y) - T(y)$

$$\begin{bmatrix}\Delta V_1\\ \Delta V_2\\ \Delta V_3\end{bmatrix} = \underbrace{\begin{bmatrix} V_{1,1} & V_{1,2} & V_{1,3}\\ V_{2,1} & V_{2,2} & V_{2,3}\\ V_{3,1} & V_{3,2} & V_{3,3}\end{bmatrix}}_{\nabla V \quad \text{2nd order tensor}} \begin{bmatrix}\Delta y_1\\ \Delta y_2\\ \Delta y_3\end{bmatrix}$$



Curvilinear orthonormal coordinate systems:

Example: polar coordinate system

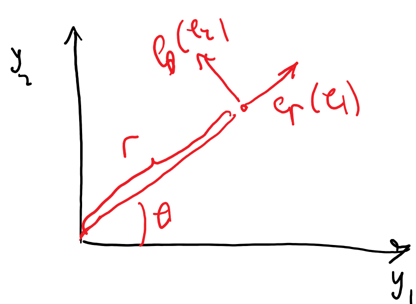
Cartesian coordinate

(y_1, y_2)

Curvilinear coordinate

(x_1, x_2) e.g. $x_1 = r$

$x_2 = \theta$



In general we express y_1, y_2 in terms of x_1, x_2

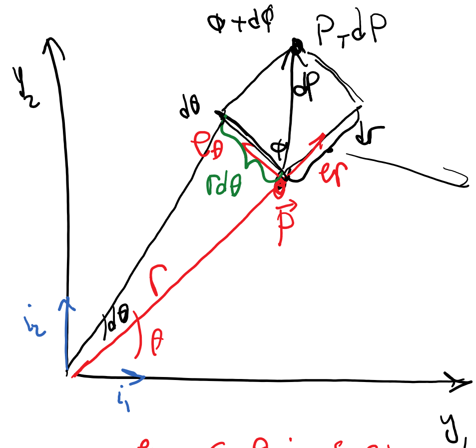
$$y_1 = r\cos\theta = x_1 \cos x_2$$
$$y_2 = r\sin\theta = x_1 \sin x_2$$

How to express gradient in polar coordinate system

$\vec{P}$ is position vector

$$\vec{P} = r e_r(\theta)$$
$$dP = dr\, e_r + r\, de_r$$
$$de_r = \frac{\partial e_r}{\partial r} dr + \frac{\partial e_r}{\partial \theta} d\theta$$

$$\boxed{dP = \underbrace{dr}_{(dP)_r}\, e_r + \underbrace{(r\, d\theta)}_{(dP)_\theta}\, e_\theta}$$



$$e_r = \cos\theta\, i_1 + \sin\theta\, i_2$$
$$e_\theta = -\sin\theta\, i_1 + \cos\theta\, i_2$$
$$\frac{\partial e_r}{\partial r} = 0, \quad \frac{\partial e_r}{\partial \theta} = -\sin\theta\, i_1 + \cos\theta\, i_2 = e_\theta$$
$$\frac{\partial e_\theta}{\partial r} = 0, \quad \frac{\partial e_\theta}{\partial \theta} = -e_r$$

$\phi(r, \theta)$

scalar

$$\underbrace{d\phi}_{\text{increment of } \phi} = \frac{\partial \phi}{\partial r} dr + \frac{\partial \phi}{\partial \theta} d\theta$$

$$d\phi = \nabla\phi \; dP \quad \text{all in polar coordinate system}$$

$$dP = (dP)_r\, e_r + (dP)_\theta\, e_\theta \qquad dP_r = dr, \quad dP_\theta = r\, d\theta$$

$$d\phi = \frac{\partial \phi}{\partial r} dr + \frac{1}{r}\frac{\partial \phi}{\partial \theta} (r\, d\theta)$$

$$\boxed{d\phi = \begin{bmatrix} \dfrac{\partial \phi}{\partial r} & \dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta} \end{bmatrix} \begin{bmatrix} dP_r \\ dP_\theta \end{bmatrix}}$$

$\nabla\phi$ expressed in polar coordinate

$$\nabla\phi = \begin{bmatrix} \dfrac{\partial \phi}{\partial r} & \dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta} \end{bmatrix}$$

Gradient of a vector in polar coordinate:

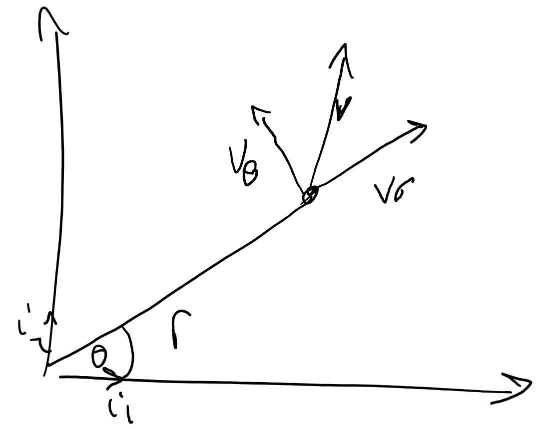
$V = V_r e_r + V_\theta e_\theta$

$dV = dV_r\, e_r + V_r\, de_r + dV_\theta\, e_\theta + V_\theta\, de_\theta$

$= (V_{r,r}\, dr + V_{r,\theta}\, d\theta)\, e_r + V_r\,(e_\theta\, d\theta)$

$+ (V_{\theta,r}\, dr + V_{\theta,\theta}\, d\theta)\, e_\theta$

$+ V_\theta\,(-e_r\, d\theta)$



$e_r = \cos\theta\, i_1 + \sin\theta\, i_2 \qquad \frac{\partial e_r}{\partial \theta} = e_\theta$

$e_\theta = -\sin\theta\, i_1 + \cos\theta\, i_2 \qquad \frac{\partial e_\theta}{\partial \theta} = -e_r$

$$\boxed{\Rightarrow \begin{bmatrix} dV_r \\ dV_\theta \end{bmatrix} = \underbrace{\begin{bmatrix} V_{r,r} & \frac{V_{r,\theta} - V_\theta}{r} \\ V_{\theta,r} & \frac{V_{\theta,\theta} + V_r}{r} \end{bmatrix}}_{\nabla V \text{ in polar coordinate}} \begin{bmatrix} dP_r \to dr \\ dP_\theta \to r\,d\theta \end{bmatrix}}$$