CM2021/10/07

Tuesday, October 5, 2021 4:31 PM

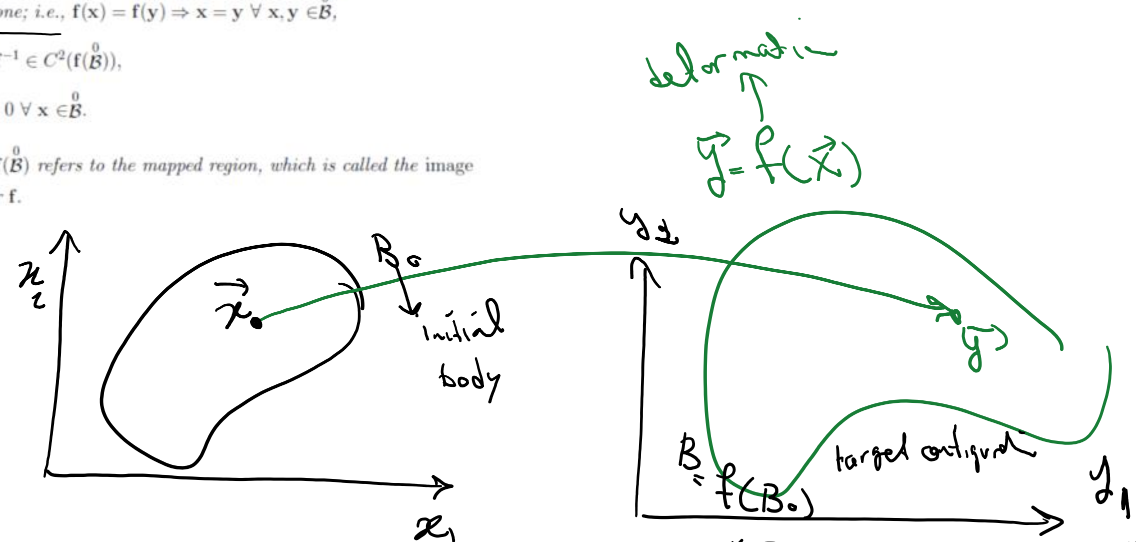
Kinematics:

$\vec{n}$

Definition 72 *Let $\overset{0}{\mathcal{B}}$ be an open, bounded, regular region of a Euclidean point space $\mathcal{E}$. A deformation* $\mathbf{f}$ *is a mapping (function) of points in $\overset{0}{\mathcal{B}}$ onto another open region of $\mathcal{E}$ with the properties*

1. $\mathbf{f}$ *is one-to-one; i.e.,* $\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{y}) \Rightarrow \mathbf{x} = \mathbf{y} \ \forall \ \mathbf{x}, \mathbf{y} \in \overset{0}{\mathcal{B}}$,
2. $\mathbf{f} \in C^2(\overset{0}{\mathcal{B}})$, $\mathbf{f}^{-1} \in C^2(\mathbf{f}(\overset{0}{\mathcal{B}}))$,
3. $\det \nabla \mathbf{f}(\mathbf{x}) > 0 \ \forall \ \mathbf{x} \in \overset{0}{\mathcal{B}}$.

The notation $\mathbf{f}(\overset{0}{\mathcal{B}})$ refers to the mapped region, which is called the image of the set $\overset{0}{\mathcal{B}}$ under $\mathbf{f}$.

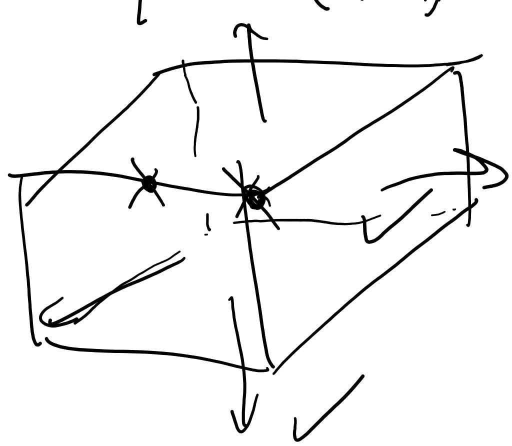


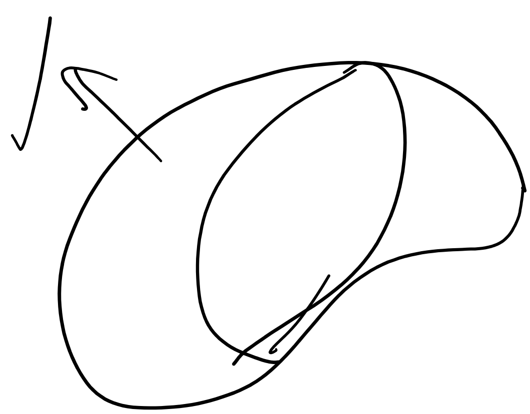
Reference configuration "Current configuration"

" undeformed configuration " (Solid Mechanics) → " deformed configuration "

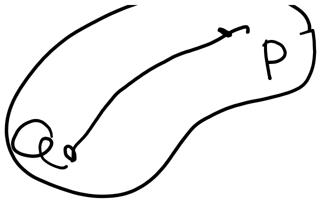
spatial configuration

Requirement: " nice boundaries " that normal vector can be defined almost every where (a.e.)





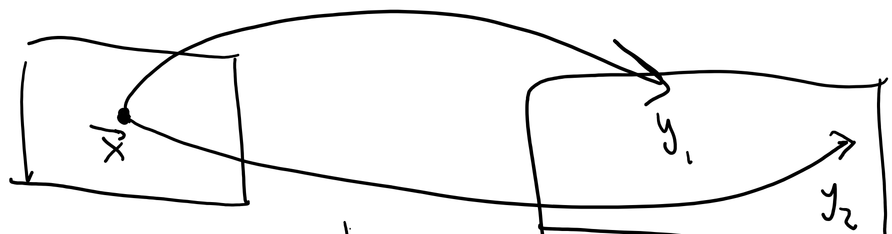
Region : it's simply connected



Properties of f:

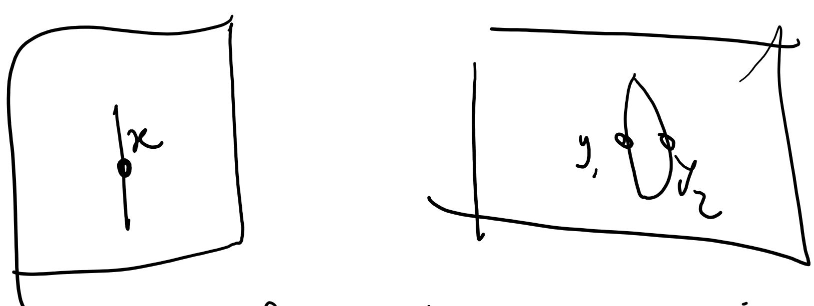
1. f is a 1-1 function

function



f(x) is uniquely defined

How about fracture

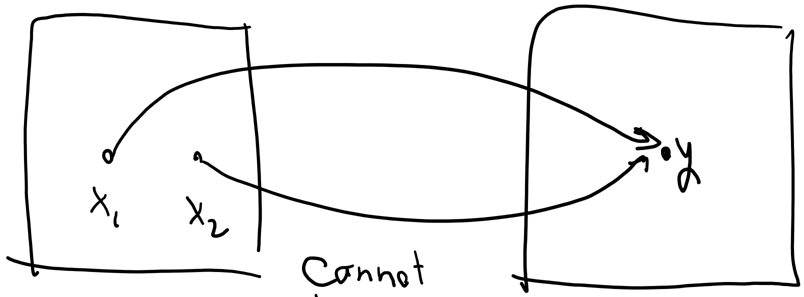


this is fine because x is on the boundary of B

(∂B) not included in B°

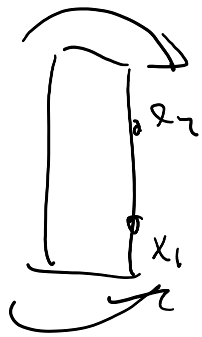
boundary of B

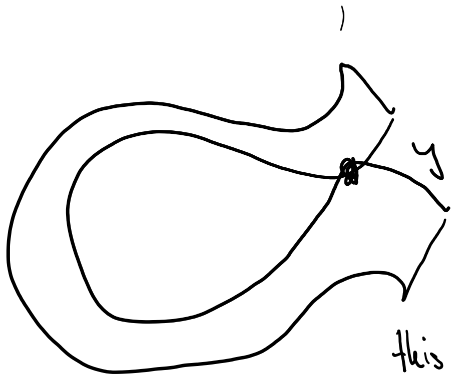
1-1 property



Cannot happen

How about this





$y = f(x_1) = f(x_2)$

this is fine

because ~~x~~x_1 & x_2 are not inside B

Property 2:

open set of B

f must be $C^2(\mathring{B})$

it should have 2 derivatives and f, $\frac{\partial f}{\partial x_i}$, $\frac{\partial^2 f}{\partial x_i \partial x_j}$ are continuous.

Why?

1D $\quad u = y - x$ (displacement)

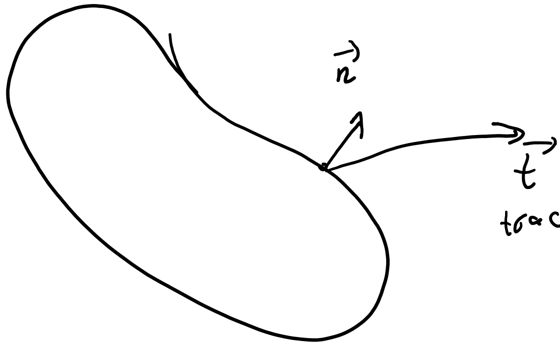
$\varepsilon = u_{,x} = y_{,x} - 1$

EOM $\quad \rho \ddot{u} = \sigma_{,x} + \rho b$

$\sigma = E\varepsilon$

$\left(E(y_{,x} - 1)\right)_{,x}$

we need to derivation



traction $\vec{t} = \sigma \cdot n$

$$\sigma = C\,(\nabla_x y - I)$$

for traction on the boundary we just need

1 derivative of $y_{,x}$ $f(x)$

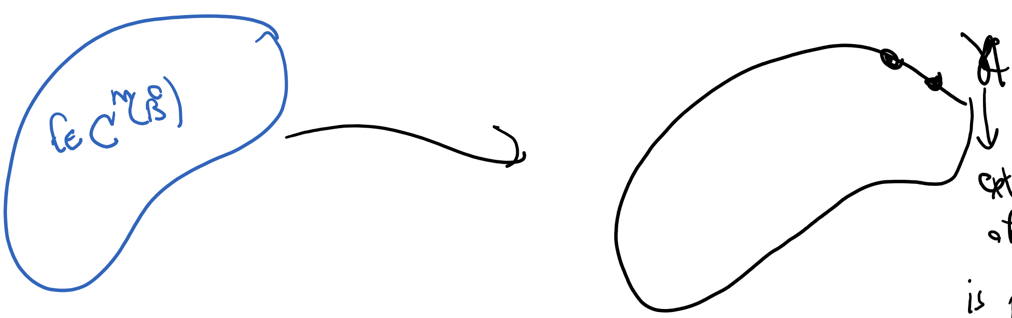
f needs 2 derivatives inside $f \in C^2(\mathring{B})$

" " 1 " on ∂B

Read Definitions 73, 74, 75 on trace operation

Trace operator: once we have a quantity defined inside a domain -> we can EXTEND it to the boundary of the domain

If a function is C^m inside the domain, we can extend it to the boundary of that domain as a C^{m-1} function.



$\gamma f = f|_{\partial B}$

extension of f is possible & is a C^{m-1} function

$$\text{if } f \in C^m(\mathring{B}) \longrightarrow \gamma f = C^{m-1}(B)$$

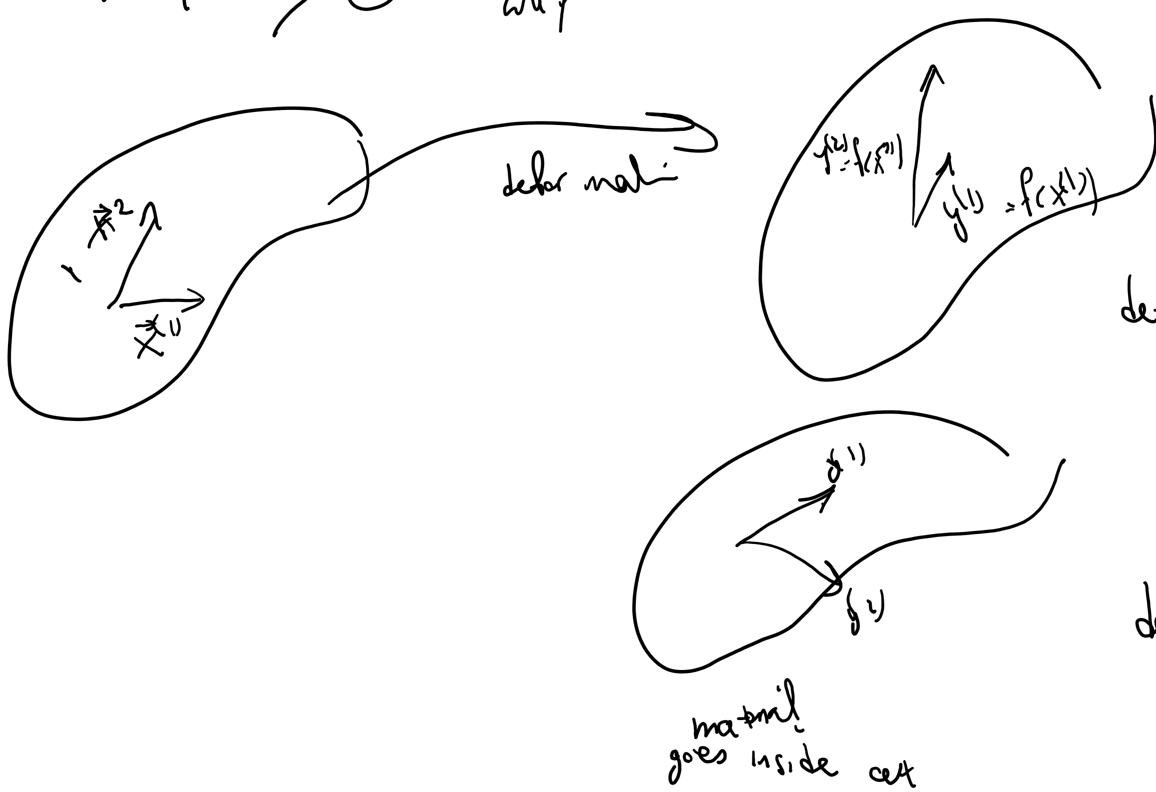
$m = 2$ $\quad y_{,ij}$ for EOM above $\quad \to$ traction on the boundary

Remark 28 *The requirement of well-behaved first partial derivatives supports the unambiguous extension of f to the boundary $\partial\mathcal{B}$. Inductively, the trace operator "evaluates" a function $f \in C^M(\overset{0}{\mathcal{B}})$ and its partial derivatives up to order $M-1$ on $\partial\mathcal{B}$. Specifically, for any deformation $\mathbf{f} \in C^2(\overset{0}{\mathcal{B}})$, the trace allows us to "evaluate" the components f_i and the partial derivatives $f_{i,j}$ on $\partial\mathcal{B}$. This is sufficient for a complete kinematic description of the closed body $\mathcal{B}$. These arguments are associated with the following Extension Theorem.*

Property 3:

$$\det \nabla f > 0$$

why



$\det \nabla f > 0$

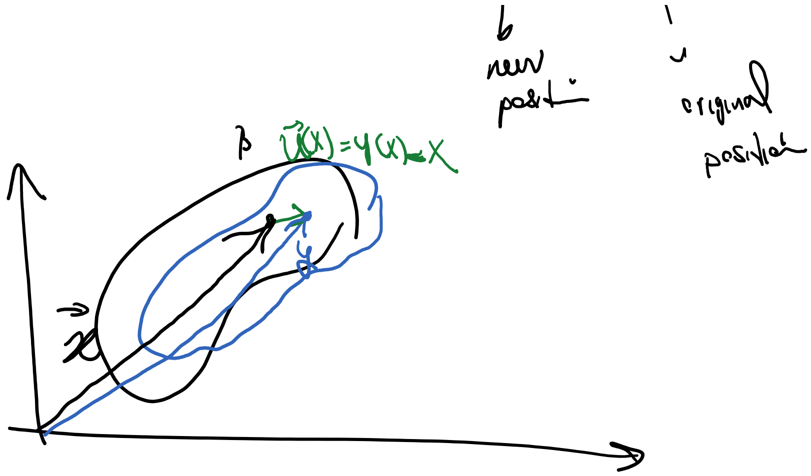
$\det \nabla f < 0$

Next, we are going to discuss
- how line segments, areas, volumes change because of deformation.
- Rigid deformation.

Definition of displacement

$$u(X) = y(X) - x$$

new position   original



u is also 1-1 & $C^2(\mathcal{B}_0)$

$$U_i(x_1, x_2, x_3) = y_i(x_1, x_2, x_3) - X_i$$

$$\underbrace{\frac{\partial u_i}{\partial x_j}}_{(\nabla u_{/x})_{ij}} = \underbrace{\frac{\partial y_i}{\partial x_j}}_{(\nabla y_{/x})_{ij}} - \frac{\partial x_i}{\partial x_j} \rightarrow \delta_{ij}$$

$$H = F - I$$

$H = \nabla u_{/x}$ $\quad H_{ij} = \frac{\partial u_i}{\partial x_j}$ $\quad$ displacement gradient

$F = \nabla y_{/x}$ $\quad F_{ij} = \frac{\partial y_i}{\partial x_j}$ $\quad$ deformation gradient

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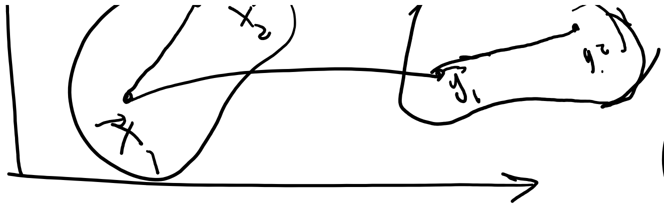
Rigid body deformation

A deformation is rigid iff it preserves distance between all pairs of points

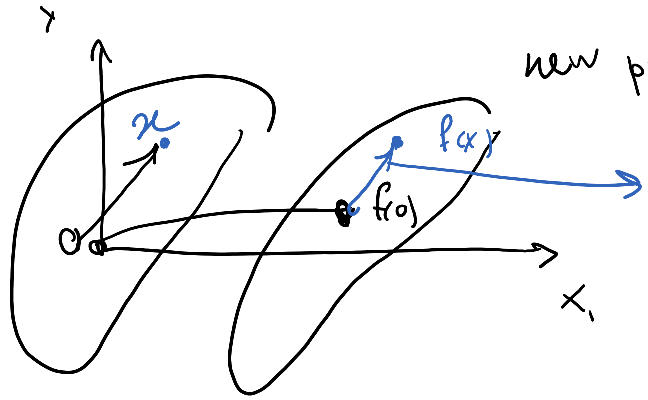


A deformation is rigid iff it preserves distance between all ~~pairs of points~~

$$\boxed{|y_2 - y_1| = |x_2 - x_1| \quad \forall\, x_1, x_2 \in \mathring{B}}$$

A Rigid body deformation consists of possible translation plus rotation

Theorem 119: Let f be a rigid deformation and O the origin. We define the relative displacement w.r.t. origin as:



$$\xi(x) = |f(x) - f(0)|$$

ξ preserves length & angle

① $|\xi(x)| = |x|$ length

② $\xi(x) \cdot \xi(y) = x \cdot y$ inner_product

$\Big\} \longrightarrow$ preserves angle

$$\underbrace{|f(x) - f(0)|}_{|\xi(x)| \,=\, |x|} = |x - 0| \qquad \text{rigid deformation} \quad (x_2 \to 0,\ x_1 \to x)$$

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property ② $\xi(x+y) \cdot \xi(x+y) = \overset{\text{we can show this}}{\xi^2(x) + 2\,\xi(x) \cdot \xi(y) + \xi^2(y)}$

& from here we show

$$\xi(x) \cdot \xi(y) = x \cdot y$$

One can show that ξ is linear

$$\xi(x) = \underbrace{Q}_{\text{orthogonal tensor}}\, x$$

orthogonal tensor

$$f(x) - f(0)$$

$$f(x) = \underbrace{f(0)}_{c} + Qx$$

translation

rotation

(why no reflection)

Rigid motion

$$f(x) = c + Qx \rightarrow \nabla_x f = Q \qquad \det \nabla_x f = \overset{\text{rotation}}{+1} \quad \cancel{-1}$$

rotation + reflection

we had the condition $\det \nabla f > 1$

Rigid motion: $|y_2 - y_1| = |x_2 - x_1|$

$$\Updownarrow$$

$$f(x) = \underset{\text{translation}}{c} + \underset{\text{rotation}}{Qx}$$

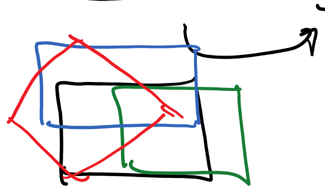
$Q^t Q = I$

$\det Q = 1$

Q : proper orthogonal

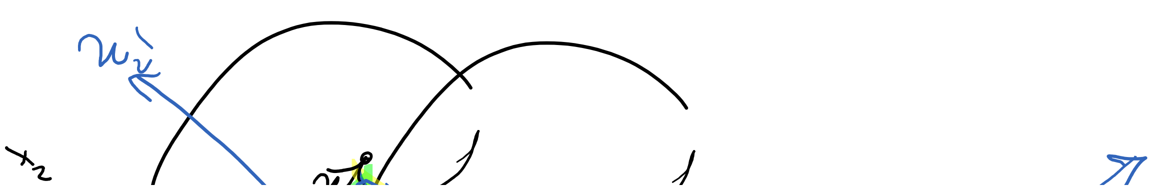
1D — 1 translation

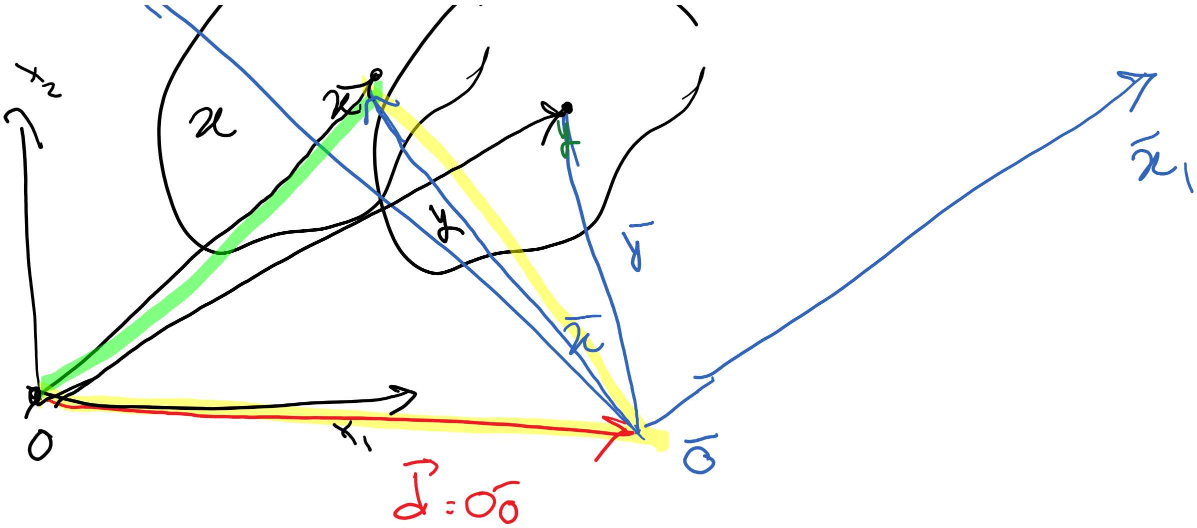
2D — 2 translations + 1 rotations



3D — 3 translations + 3 rotations

What if we represent rigid motion w.r.t. another coordinate system?





① $x = d + \bar{x}$

② $y = d + \bar{y}$

we know that deformation w.r.t. ⊗ coordinate is rigid

c, Q exist $\ni$ $y = c + Qx$

$(\bar{y} = \bar{c} + \bar{Q}\bar{x}$ need to show this$)$

$d + \bar{y} = c + Qx$

① $x = d + \bar{x}$

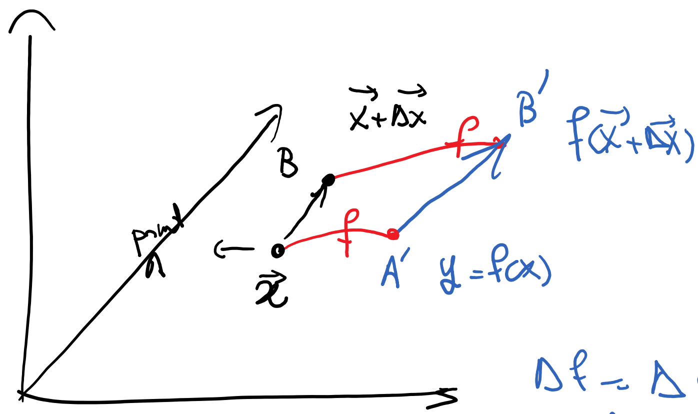
$d + \bar{y} = c + Q(\bar{x} + d)$

$$\bar{y} = \underbrace{(c + Qd - y)}_{\bar{c}} + \underbrace{Q}_{\bar{Q}}\bar{x}$$

Translation vector depends on the choice of coordinate system
BUT the rotation tensor is independent from the coordinate system

A rigid deformation is coordinate independent (if rigid deformation in one, it is rigid deformation in another one)

Change of
1D objections: line segment -> length and angle between line segments
2D objects: change of surface area
3D objects: change of volume



new segment $\vec{AB} = \underbrace{f(x+\Delta x) - f(x)}_{\Delta f = \Delta y} \simeq$

$\nabla_x f \, \Delta x + \frac{1}{2} \Delta x \, H \Delta x + \ldots$

HOT

$\nabla_{\partial x} f$ $\quad \vec{\Delta x}$

for Δx small

$$\boxed{\begin{gathered} \vec{\Delta y} \simeq F \vec{\Delta x} \quad \text{as } \vec{\Delta x} \to 0 \\ \text{2nd order tensor} \\ \text{deformation gradient} \\ F_{ij} = \frac{\partial f_i(x)}{\partial x_j} = \frac{\partial y_i}{\partial x_j} \end{gathered}}$$