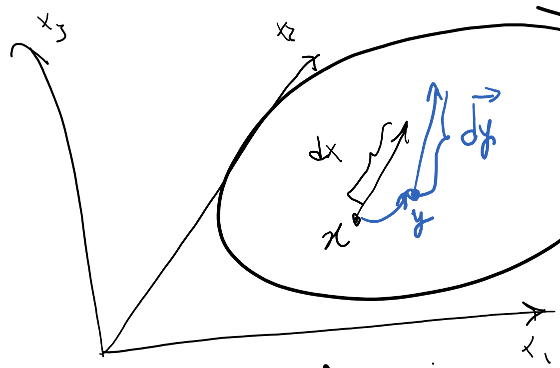
CM2021/10/12

Tuesday, October 12, 2021 4:29 PM

From last time we had

$$\vec{dy} = \underbrace{\nabla_{y/x}}_{F} \, \vec{dx}$$



Example: Let's suppose we have the following deformation

$$y_1 = 1.1x_1$$
$$y_2 = 1.5x_2$$
$$y_3 = 1.2x_1 + 1.7x_3^2$$

$$\rightarrow \quad F = \nabla_{y/x} = \begin{bmatrix} 1.1 & 0 & 0 \\ 0 & 1.5 & 0 \\ 1.2 & 0 & 3.4x_3 \end{bmatrix} \qquad F_{ij} = \frac{\partial y_i}{\partial x_j}$$

$$\vec{dy} = \quad '' \quad \vec{dx}$$

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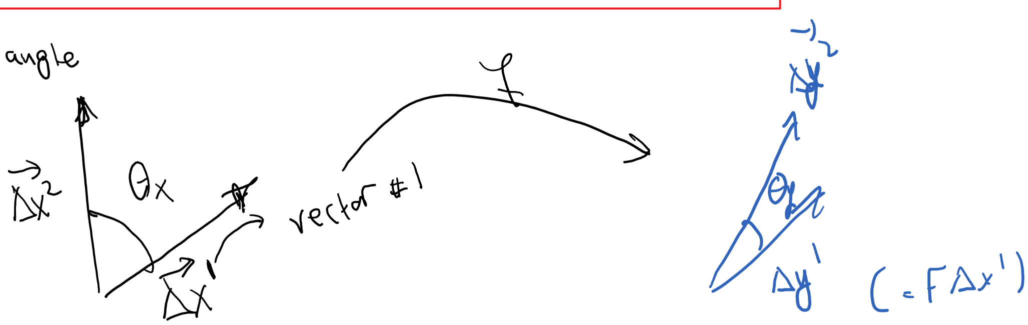
1) $|\Delta y| = ?$ $\quad |\Delta y| = \sqrt{\Delta y \cdot \Delta y} = \sqrt{(F\Delta x) \cdot (F\Delta x)}$

$$= \sqrt{F^t F \Delta x \cdot \Delta x}$$

1

$$\boxed{C = F^t F \quad \text{right Cauchy tensor} \qquad |\Delta y| = \sqrt{\Delta x \cdot C \Delta x}}$$

2) Change of angle



$\theta_y = ?$

recall $a \cdot b = |a||b| \cos\theta_{a,b} \rightarrow \cos\theta_{a,b} = \dfrac{a \cdot b}{|a||b|}$

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$$\cos\theta_y = \frac{\vec{\Delta y}^1 \cdot \vec{\Delta y}^2}{|\vec{\Delta y}^1|\,|\Delta y^2|} = \frac{F\vec{\Delta x}^1 \cdot F\Delta x^2}{\underbrace{\sqrt{\Delta x^1 \cdot C\Delta x^1}}_{|\Delta x^1|}\;\underbrace{\sqrt{\Delta x^2 \cdot C\Delta x^2}}_{|\Delta x^2|}}$$

from eq 4 above

$$\cos\theta_y = \frac{\overbrace{F^tF}^{C}\,\Delta x^1 \cdot \Delta x^2}{\sqrt{\Delta x^1 \cdot C\Delta x^1}\,\sqrt{\Delta x^2 \cdot C\Delta x^2}}$$

(2)

$$\boxed{\cos\theta_y = \frac{\vec{\Delta x}_1 \cdot C\,\vec{\Delta x}_2}{\sqrt{\Delta x^1 \cdot C\Delta x^1}\,\sqrt{\Delta x^2 \cdot C\Delta x^2}}}$$

$\vec{\Delta x}_1 \cdot C\,\vec{\Delta x}_2 = \vec{\Delta x}_2 \cdot C\Delta x_1$

$C = F^tF$ Right Cauchy-Green strain tensor

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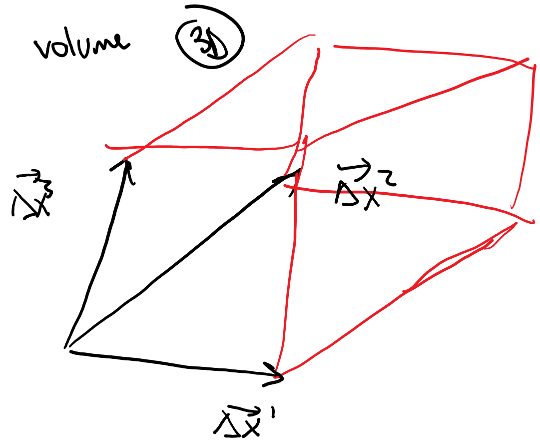
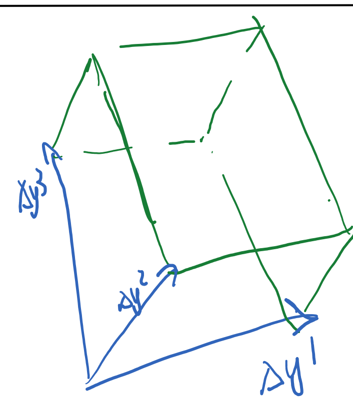
3) Change of volume (3D)





$$\Delta x^1 = \begin{bmatrix} \Delta x_1^1 \\ \Delta x_2^1 \\ \Delta x_3^1 \end{bmatrix}$$

$\Delta x_2^1 \rightarrow$ component 2 of Δx^1

$$\Delta V_x = (\Delta x^1 \times \Delta x^2) \cdot \Delta x^3$$

$$\Delta V_y = (\Delta y^1 \times \Delta y^2) \cdot \Delta y^3$$

$$= \det \begin{bmatrix} \Delta x_1^1 & \Delta x_2^1 & \Delta x_3 \\ \Delta x_1^2 & \Delta x_2^2 & \Delta x_3^2 \\ \Delta x_1^3 & \Delta x_2^3 & \Delta x_3^3 \end{bmatrix} \begin{array}{l} \rightarrow \Delta x^1 \\ \rightarrow \Delta x^2 \\ \rightarrow \Delta x^3 \end{array}$$

This is the situation we deal with in change of integral variable

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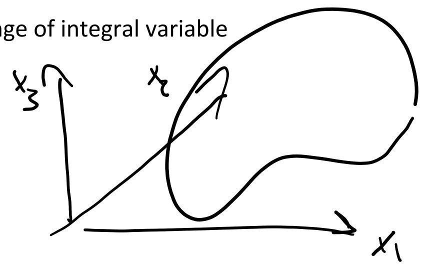
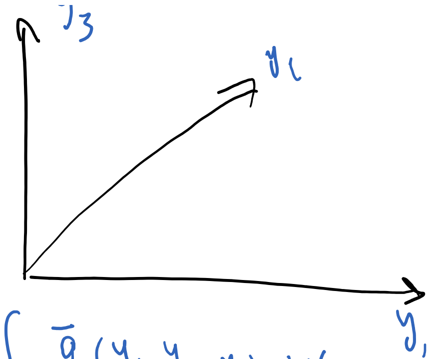




$$I_g^x = \int_{D_x} g(x_1, x_2, x_3)\, dV_x$$

$$I_g^y = \int_{D_y} \bar{g}(y_1, y_2, y_3)\, dV_y$$

change variable from y to x

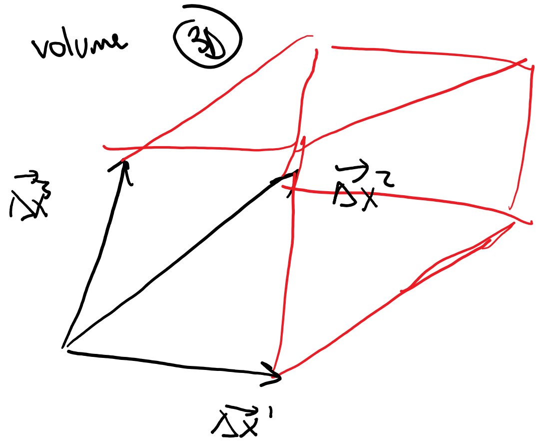
$$\int_{D_x} \bar{g}\left(y_1(\vec{x}), y_2(\vec{x}), y_3(x)\right) dV_x$$

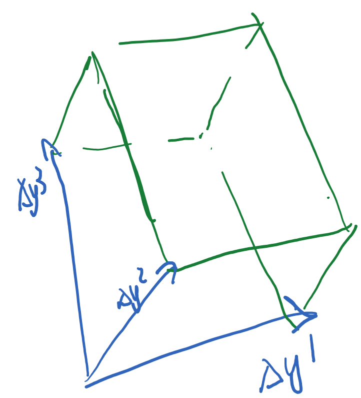
$$dV_y = \underbrace{|J|}_{\det} dV_x$$

$$J = \nabla y_{/x}$$

We have seen this relation in calculus. I'm going to prove it here for the continuum mechanics application of change of volume.

3] Change of volume (3D)





$$\Delta x^1 = \begin{bmatrix} \Delta x^1_1 \\ \Delta x^1_2 \\ \Delta x^1_3 \end{bmatrix}$$

$\Delta x^1_2 \rightarrow$ component 2 of Δx^1

$$\Delta V_x = (\Delta x^1 \times \Delta x^2) \cdot \Delta x^3$$

$$\Delta V_y = (\Delta y^1 \times \Delta y^2) \cdot \Delta y^3$$

$$\Delta V_y = \ ? \ \Delta V_x$$

$$\Delta y^M = F \Delta x^m$$

$$\Delta y^m = F\Delta x^m$$

$$\underline{\Delta V_y} = \epsilon_{ijk}\, \Delta y^1_i\, \Delta y^2_j\, \Delta y^3_k$$

theorem

$$(Fa)\times(Fb)\cdot Fc = \det F\; a\times b\cdot c$$

why?

$$(Fa)\times(Fb)\cdot Fc = \epsilon_{ijk}\,(Fa)_i\,(Fb)_j\,(Fc)_k$$
$$= \epsilon_{ijk}\,(F_{im}a_m)(F_{jn}b_n)(F_{kp}c_p)$$
$$= \underbrace{(\epsilon_{ijk}\, F_{im}\, F_{jn}\, F_{kp})}\, a_m b_n c_p$$
$$(\epsilon_{mnp} \det F)\, a_m b_n c_p = \det F\,(\underbrace{\epsilon_{mnp}\, a_m b_n c_p})$$
$$= \det F\,(a\times b)\cdot c$$

Now Let $a = \Delta x^1$

$b = \Delta x^2$

$c = \Delta x^3$

$$((F\Delta x^1)\times(F\Delta x^2))\cdot F\Delta x^3 = \det F\,(\underbrace{\Delta x^1\times\Delta x^2)\cdot\Delta x^3}_{\Delta V_x}$$
$$\underbrace{(\Delta y^1\times\Delta y^2)\cdot\Delta y^3}_{\Delta V_y} = \det F \quad \Delta V_x$$

(3)

$$\boxed{\begin{gathered}\Delta V_y = \det F\,\Delta V_x \\ F = \nabla y_x\end{gathered}}$$

why we wanted to have $\det F > 0$

Definition 72 *Let $\overset{0}{\mathcal{B}}$ be an open, bounded, regular region of a Euclidean point space $\mathcal{E}$. A* deformation $\mathbf{f}$ *is a mapping (function) of points in $\overset{0}{\mathcal{B}}$ onto another open region of $\mathcal{E}$ with the properties*

1. $\mathbf{f}$ *is one-to-one; i.e.,* $\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{y}) \Rightarrow \mathbf{x} = \mathbf{y}\ \forall\ \mathbf{x}, \mathbf{y} \in \overset{0}{\mathcal{B}}$,
2. $\mathbf{f} \in C^2(\overset{0}{\mathcal{B}})$, $\mathbf{f}^{-1} \in C^2(\mathbf{f}(\overset{0}{\mathcal{B}}))$,

2. $\mathbf{f} \in C^2(\overset{0}{\mathcal{B}})$, $\mathbf{f}^{-1} \in C^2(\mathbf{f}(\overset{0}{\mathcal{B}}))$,

3. $\det \nabla \mathbf{f}(\mathbf{x}) > 0 \ \forall \ \mathbf{x} \in \overset{0}{\mathcal{B}}$.

$\Delta V_x > 0 \rightarrow \Delta V_y > 0$

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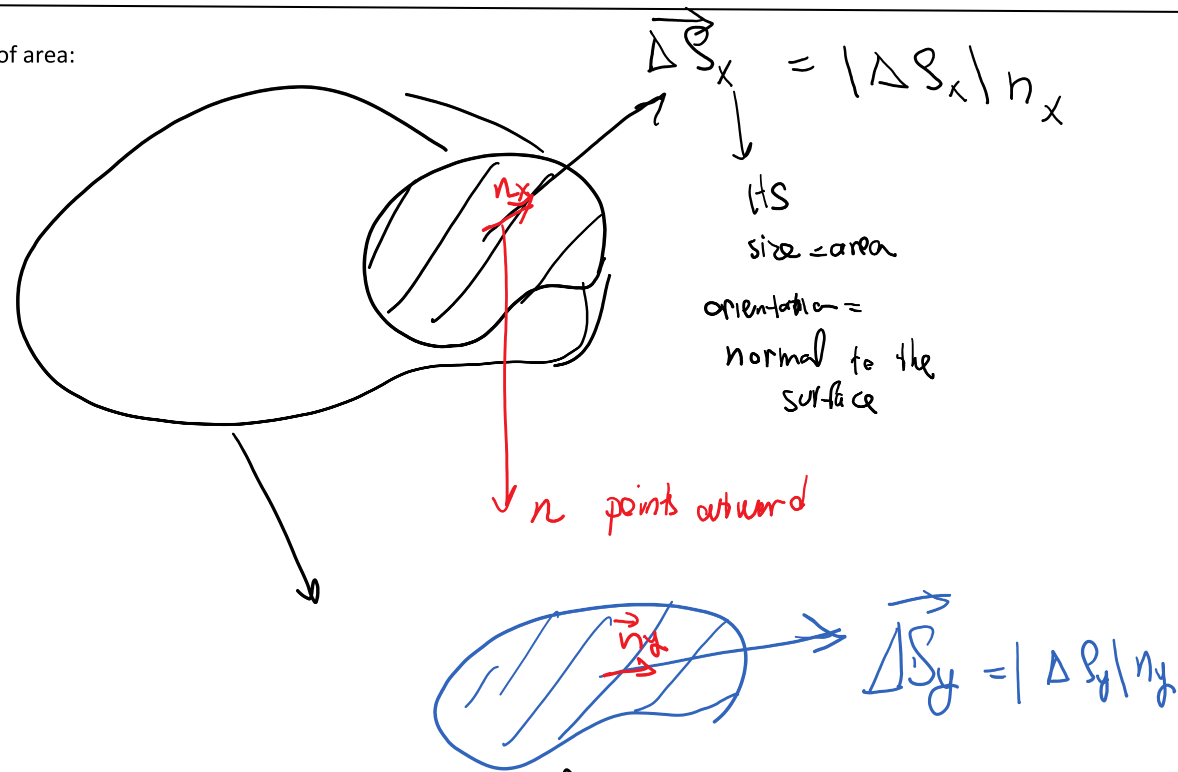
4. Change of area:



both the orientation & magnitude of normal vector change in general

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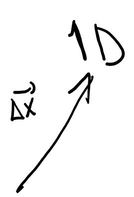
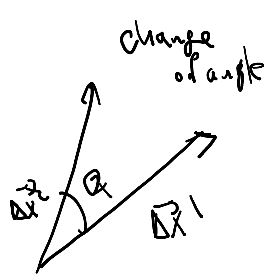
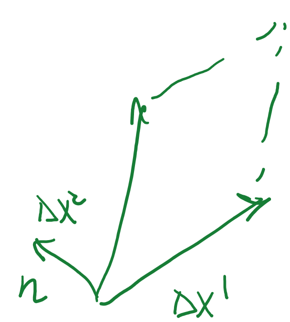
How do we calculate the change of area?



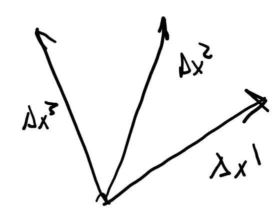
$\vec{\Delta y} = F \Delta x$

change of length

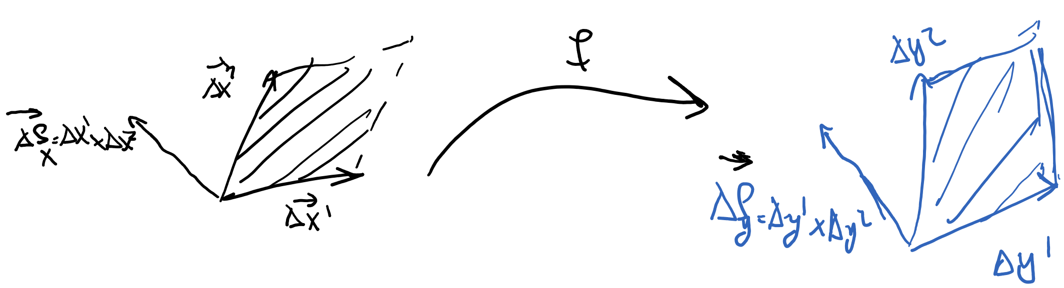




$dS_x = \vec{\Delta x}^1 \times \vec{\Delta x}^2$



Change of volume



$$\underbrace{\Delta \vec{S}_y}_{\text{vector}} = \underbrace{\text{?}}_{\text{2nd order tensor}} \; \underbrace{\Delta \vec{S}_x}_{\text{vector}}$$

$$\Delta\vec{S}_y = (\Delta S_y)_1 e_1 + (\Delta S_y)_2 e_2 + (\Delta S_y)_3 e_3$$

$$(\Delta S_y)_i = (\Delta \vec{S}_y)\cdot e_i = (\Delta y^1 \otimes \Delta y^2)\cdot e_i \qquad (Fa \times Fb)\cdot Fc = \det F\, a \times b \cdot c$$

$$= (F\Delta x^1 \times F \Delta x^2) \cdot \underbrace{e_i}_{FF^{-1}e_i}$$

$$= (F\underbrace{\Delta x^1}_{a} \times F\underbrace{\Delta x^2}_{b}) \cdot F(\underbrace{F^{-1}e_i}_{c}) = \det F\,(a \times b \cdot c) = \det F\,(\Delta x^1 \times \Delta x^2 \cdot F^{-1} e_i)$$

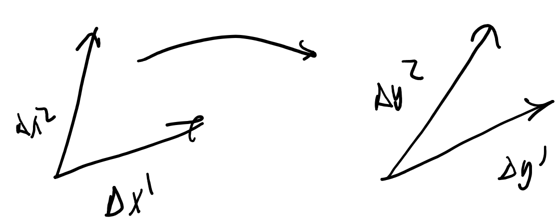
$$(F^{-1})^t = F^{-t}$$

$$\det F\; F^{-t}\, \underbrace{\Delta x^1 \times \Delta x^2}_{\Delta \vec{S}_x} \cdot e_i] \Longrightarrow$$

$$(\Delta S_y)_i = \det F\, (F^{-t} \Delta S_x)_i \longrightarrow$$

$$\boxed{\Delta \vec{S}_y = \underbrace{(\det F\; F^{-t})}_{\text{2nd order tensor mapping area vectors}} \Delta \vec{S}_x}$$

Another proof of this:



$$\Delta \vec{S}_y = \Delta y^1 \times \Delta y^2 = e_{ijk}\, \Delta y^1_i\, \Delta y^2_j\, e_k \longrightarrow$$

$$\Delta y^1 = F \Delta x^1 \Rightarrow \Delta y^1_i = F_{im} \Delta x^1_m$$

$$\Delta y^2 = F \Delta x^2 \Rightarrow \Delta y^2_j = F_{jn} \Delta x^2_n$$

$$\Delta y^2 = F\Delta x^2 \Rightarrow \Delta y^2_j = F_{jn}\,\Delta x^2_n$$

$$\Delta S_y = \epsilon_{ijk}\,\underbrace{(F_{im}\,\Delta x^1_m)}_{\Delta y^1_i}\,\underbrace{(F_{jn}\,\Delta x^2_n)}_{\Delta y^2_j}\,e_k =$$

$$\frac{\det F}{\det F}\,\big(\epsilon_{ijk}\,F_{im}\,F_{jn}\big)\,\Delta x^1_m\,\Delta x^2_n\,e_k$$

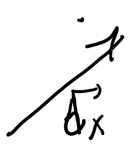
$$\epsilon_{ijk}\,F_{im}\,F_{jn} = \det F\,\epsilon_{mnp}\,F^{-1}_{pk} \quad \rightarrow$$

$$\Delta S_y = \big(\epsilon_{mnp}\,F^{-1}_{pk}\det F\big)\,\Delta x^1_m\,\Delta x^2_n\,e_k = \det F\,\underbrace{\big(\epsilon_{mnp}\,\Delta x^1_m\,\Delta x^2_n\big)}_{(\Delta x^1 \times \Delta x^2)_p}\,F^{-1}_{pk}\,e_k$$

$$= \det F\; F^{-T}_{kp}\,(\Delta x^1 \times \Delta x^2)_p\; e_k$$

$$= \det F\,\big(F^{-T}\Delta S_x\big)_k\,e_k = \det F\,F^{-T}\,\Delta \vec{S}_x$$

Summary of all these relations

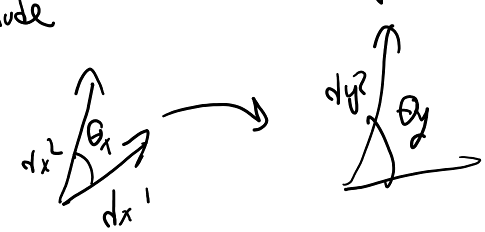


$\vec{dy} = F\,dx$, $F = \nabla_{y/x}$

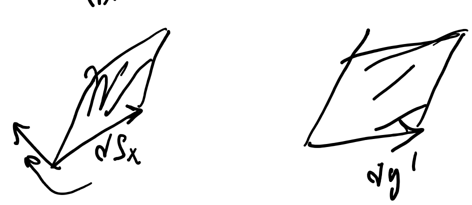
$C = F^t F$ right Cauchy ~~Green~~ tensor

$|\vec{dy}| = \sqrt{dx \cdot C\,dx} = |U\,dx|$ magnitude

$$\cos\theta_y = \frac{dx^1 \cdot C\,dx^2}{\sqrt{dx^1 \cdot C\,dx^1}\,\sqrt{dx^2 \cdot C\,dx^2}}$$



area $\quad d\vec{S}_y = \det F\;F^{-t}\,dS_x$



volume $\quad dV_y = \det F \, dV_x$

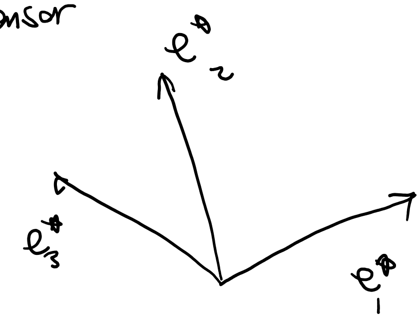
Understanding the effect of C

$$C = F^t F$$

right deformation tensor (right Cauchy Green)

positive definite

symmetric



e^*_i are orthonormal eigen vectors

$$[C]^* = \begin{bmatrix} C^*_{11} & 0 & 0 \\ 0 & C^*_{22} & 0 \\ 0 & 0 & C^*_{33} \end{bmatrix}$$

C^*_{ii} eigenvalues corresponding to e^*_i eigenvector

no summation

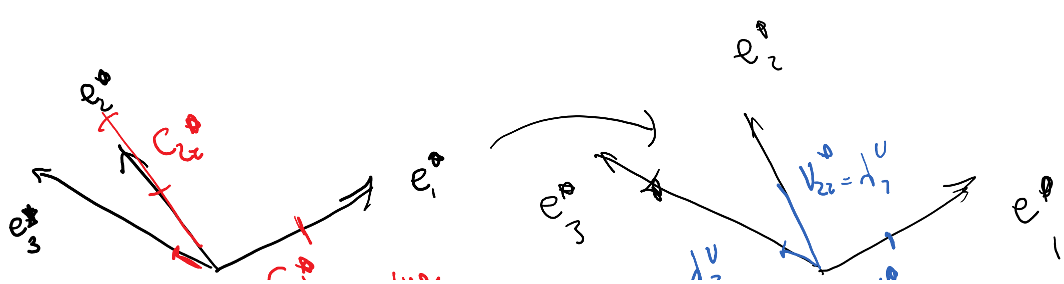
$$U = \sqrt{C}$$

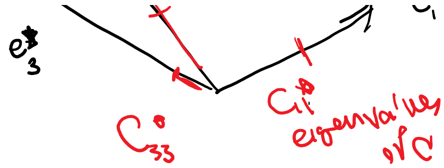
$$[U]^* = \begin{bmatrix} +\sqrt{C^*_{11}} & 0 & 0 \\ 0 & +\sqrt{C^*_{22}} & 0 \\ 0 & 0 & +\sqrt{C^*_{33}} \end{bmatrix} \qquad U^*_{ii} = \sqrt{C^*_{ii}} \quad \text{no summation}$$

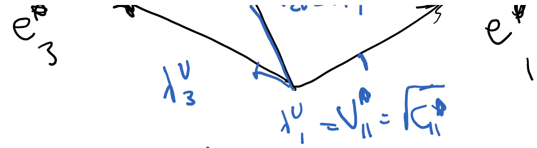
$$|\vec{dy}| = \sqrt{dx \cdot C dx} = \sqrt{dx \cdot U^2 dx} = \sqrt{dx \cdot U(U dx)}$$

$$= \sqrt{U^t dx \cdot U dx} = \sqrt{U dx \cdot U dx} = |U dx|$$

$U = U^t$ sym







$$[U]^* = \begin{bmatrix} \lambda_1^U & 0 & 0 \\ 0 & \lambda_2^U & 0 \\ 0 & 0 & \lambda_3^U \end{bmatrix} \qquad \lambda_i^U = \sqrt{\lambda_i^C}$$

$\vec{dx} = e_1^* \, d\lambda$

$\rightarrow |\vec{dy}| = ?$

$$dy = |U \vec{dx}| = |\lambda_1^U \vec{dx}| = \lambda_1^U \, |dx|$$
$\lambda_1^U > 0$

U is called $\underbrace{\quad}_{\text{right}}$ stretch tensor