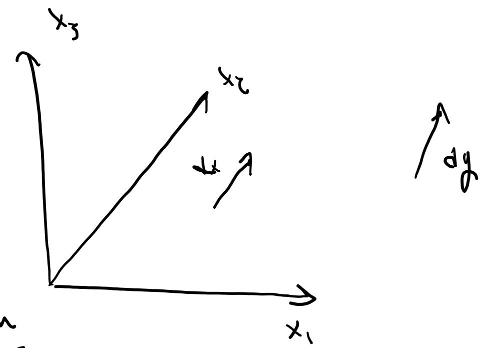
Recall from last time

$dy = F\,dx$

$|dy| = \sqrt{dx \cdot C\,dx} = |U\,dx|$



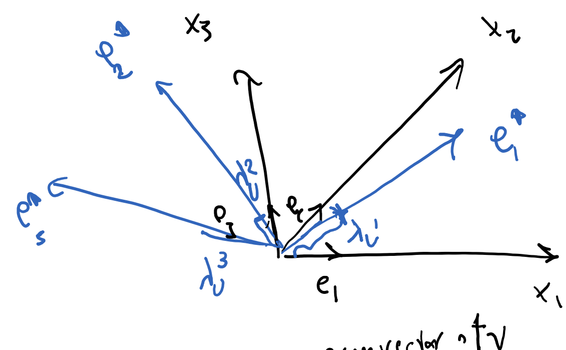
$C = F^t F$, $U = \sqrt{C}$ right stretch tensor

C → right Cauchy-Green deformation tensor

C is sym, p.s. def ⇒ has three >0 eigen values and orthogonal eigenvectors

$[C]^* = \begin{bmatrix} \lambda_C^1 & 0 & 0 \\ 0 & \lambda_C^2 & 0 \\ 0 & 0 & \lambda_C^3 \end{bmatrix}$

$[U]^* = [\sqrt{C}]^* = \begin{bmatrix} \sqrt{\lambda_C^1} & 0 & 0 \\ 0 & \sqrt{\lambda_C^2} & 0 \\ 0 & 0 & \sqrt{\lambda_C^3} \end{bmatrix}$



if $dx = |dx|\, e_2^* \rightarrow |dy| = |U\,dx| = |dx|\,|U e_2^*| = |dx|\,\lambda_U^2$

$$\boxed{\frac{|dy|}{|dx|} = \lambda_U^i \quad \text{for } dx \text{ along } e_i^*}$$

$F = RU = VR$ from polar decomposition

$\det F > 0 \rightarrow R$ simply a rotation

$F = \nabla y_{/x}$

$dy = F\,dx = R\,U\,dx = \underbrace{R}(\underbrace{U\,dx})$

general deformation → $\underbrace{\text{stretch part}}_{\text{physical strain}}$ + $\underbrace{\text{rotation}}_{\text{rotation}}$ + translation

compare this with rigid deformation

Rigid deformation $\quad y = \underbrace{Q}_{\text{rotation}}x + \underbrace{c}_{\text{translation}} \rightarrow$

$$F = \nabla y_{/x} = \underbrace{Q}_{\text{rotation}} = Q\underbrace{I}_{U} = \underbrace{I}_{V}Q$$

rigid motion we just have rotation(Q) + translation but no stretching

$U = V = I$

$$C = F^t F = Q^t Q = I \rightarrow U = I$$

$$B = F F^t = Q Q^t = I \rightarrow V = I$$

Comparing this to general deformation:

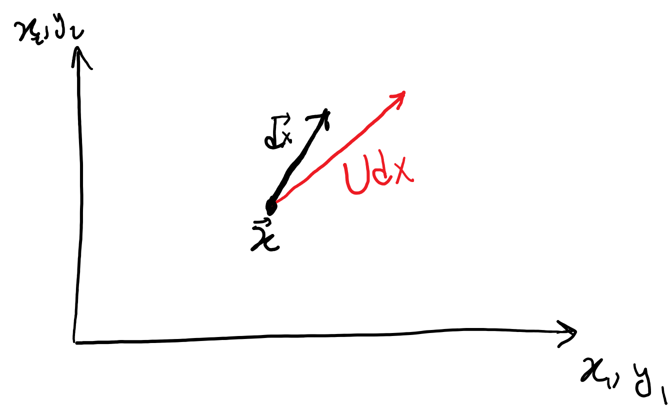
1. Stretching
2. Rotation
3. Translation

↓ right path $\quad$ ↑ Left path

Right patch

1) Stretch

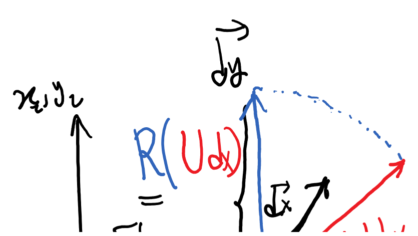
$U\,dx$



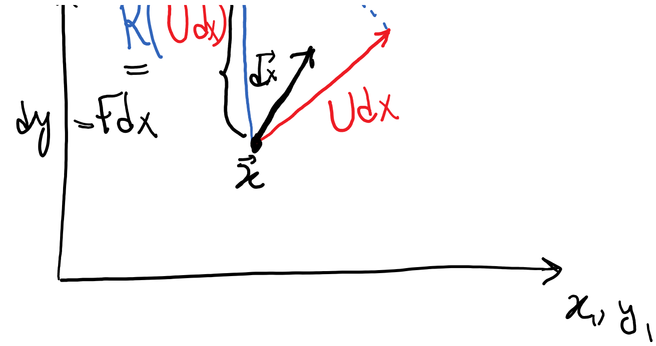
when $dx = |dx| e_1 \rightarrow$
$U dx = \lambda_j^i dx$

2) Rotation $\quad R$

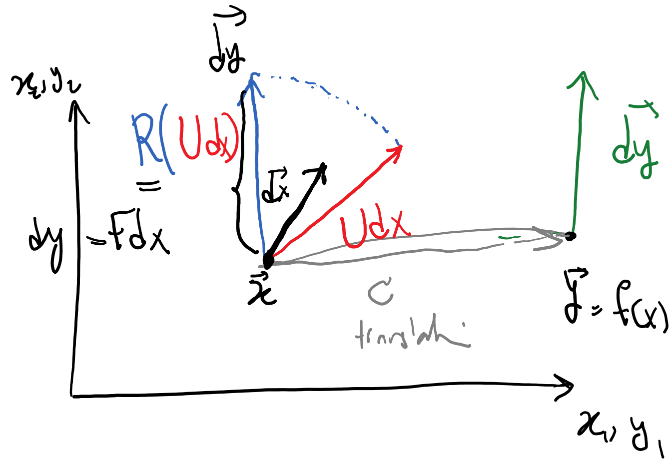
$$dy = F\,dx = R U dx$$



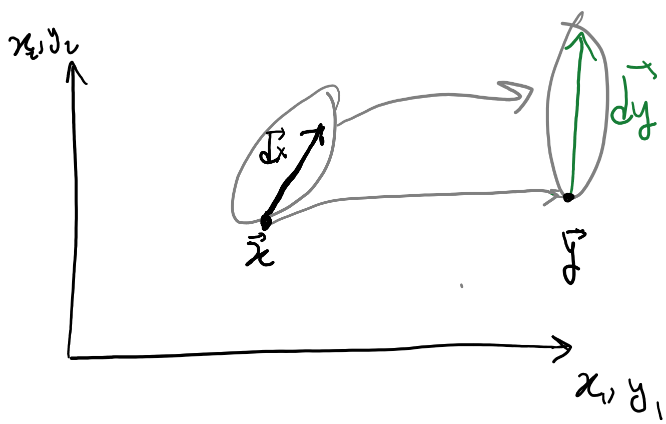
$dy = F\,dx = RU\,dx$



3) Rigid translation



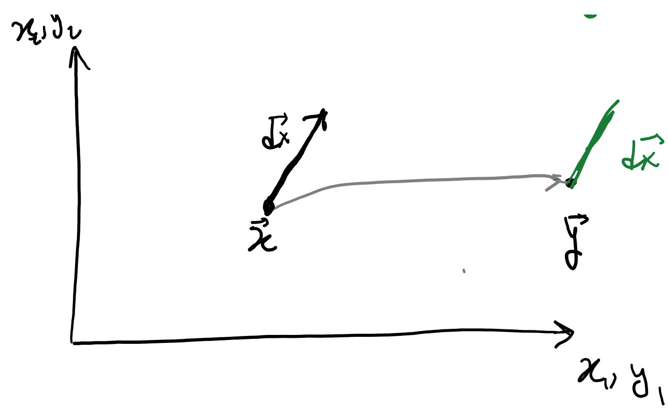
Summary:



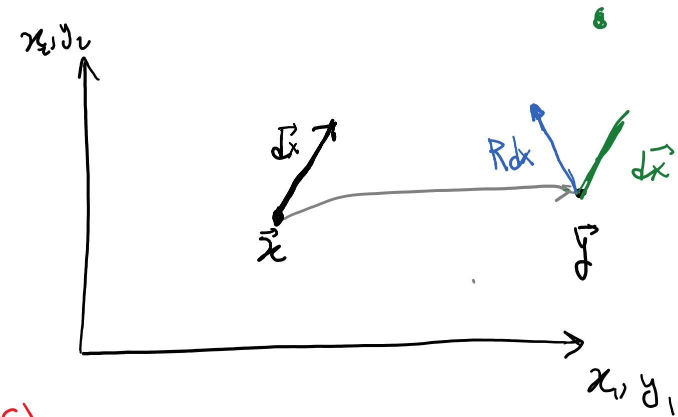
Left path

stretching $\underset{\longleftarrow}{\overset{\text{Right}}{\longrightarrow}}$ rotation $\underset{\text{Left}}{\overset{\longrightarrow}{\longleftarrow}}$ translation

1. translation



2. Rotation

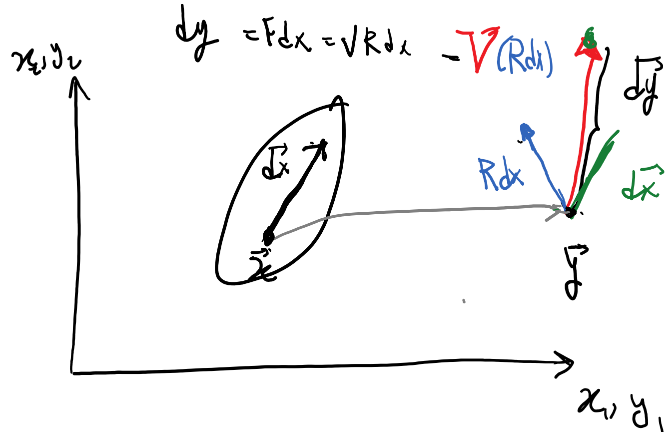


3. Stretching (V left stretch tensor)

$dy = Fdx = VRdx = V(Rdx)$

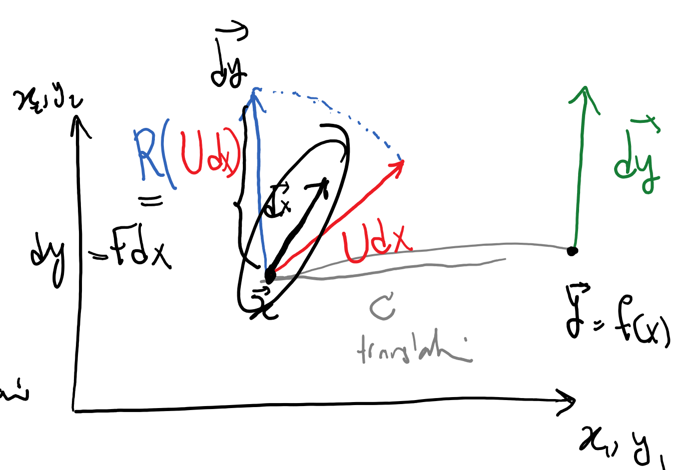
all left path

Convenient with Lagrangian representation



all right path

Convenient with Eulerian rep.



$F = \nabla_{y/x}$ $\qquad F = \underbrace{RU = VR}_{\text{general formulas}}$

$U \neq V$ in general

$RU \neq UR$

why?

since $AB \neq BA$

we have different stretch tensors

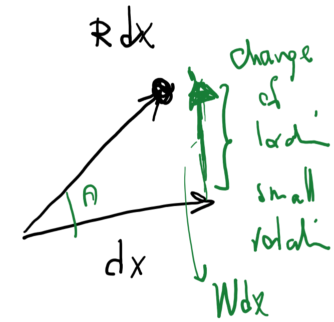
Let's compare this with small deformation gradient theory

$$H = \nabla_{u/x} = \overbrace{\left(\frac{H+H^t}{2}\right)}^{\text{sym}} + \overbrace{\left(\frac{H-H^t}{2}\right)}^{\text{skew}} \qquad u = y - x$$

$\vec{dy} - \vec{dx}$ $\qquad E \qquad W$

E → similar to U & V

W → change of location from rotation → R



$H = E + W = W + E$

summation is commutative

unlike multiplication $RU \neq UR$

Theorem 128:

1. $C = U^2$ $\qquad B = V^2$

2. $V = RUR^t$

$U = R^tVR$

3. $B = RCR^t$

$C = R^tBR$

V is the rotation of tensor U by R

U " " " " " V " R^t

B " " " " " C by R

C " " " " " B by R^t

proof

$F = RU = VR \quad \times R^{-1} \rightarrow V = RUR^{-1}, \quad R^{-1} = R^t \rightarrow V = RUR^t$

Similarly $U = R^tVR$

$$V\cdot V = (RUR^t)(RUR^t) = RU(\underbrace{R^tR}_{I})UR^t = R\underbrace{U^2}_{C}R^t = RCR^t$$

Interpretation

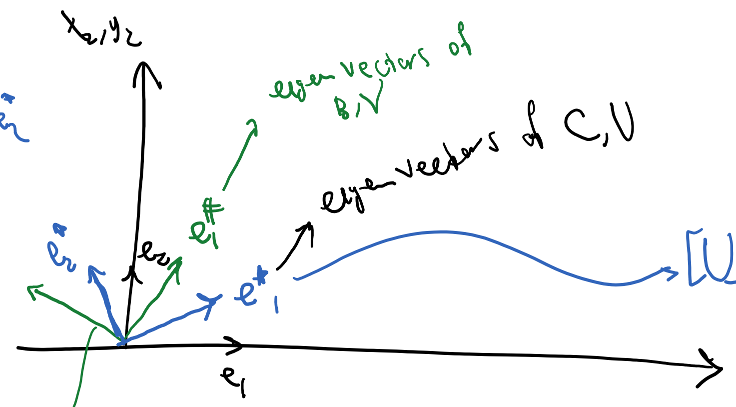
U, C have eigenvectors e^*_1, e^*_2

V, B " " $e^\#_1, e^\#_2$

$Ue^*_i = \lambda_i e^*_i$

$\downarrow$

$V(\underbrace{Re^*_i}_{e^\#_i}) = \lambda_i(Re^*_i)$



$$[U]^* = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$[V]^\# = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

eigenvalues of $\underset{B}{V}$ & $\underset{C}{U}$ are identical

eigenvectors of $\underset{B}{V}$ are eigenvectors of $\underset{C}{U}$ rotated by R

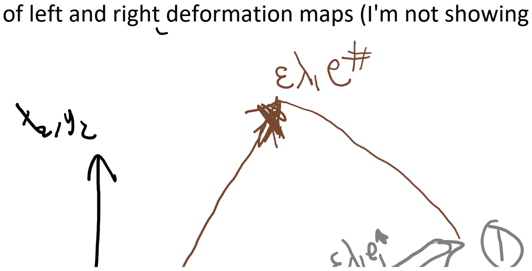
— e^*_i is eigenvector i for U $\quad Ue^*_i = \lambda_i e^*_i$ $\quad$ no summation on i

premultiply by R

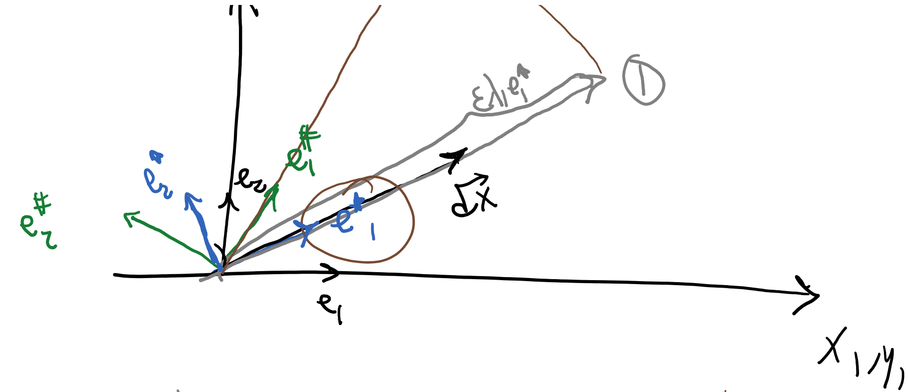
$$R U e^*_i = \lambda_i R e^*_i \qquad (\underbrace{R^tR}_{I})$$

$$(RUR^t)(Re^*_i) = \lambda_i(Re^*_i), \quad V = RUR^t \quad \rightarrow \quad V Re^*_i = \lambda_i e^*_i$$

An example of left and right deformation maps (I'm not showing translation) for a particular fiber dx $= \varepsilon e^*_1$



Right path $\qquad$ eigenvector



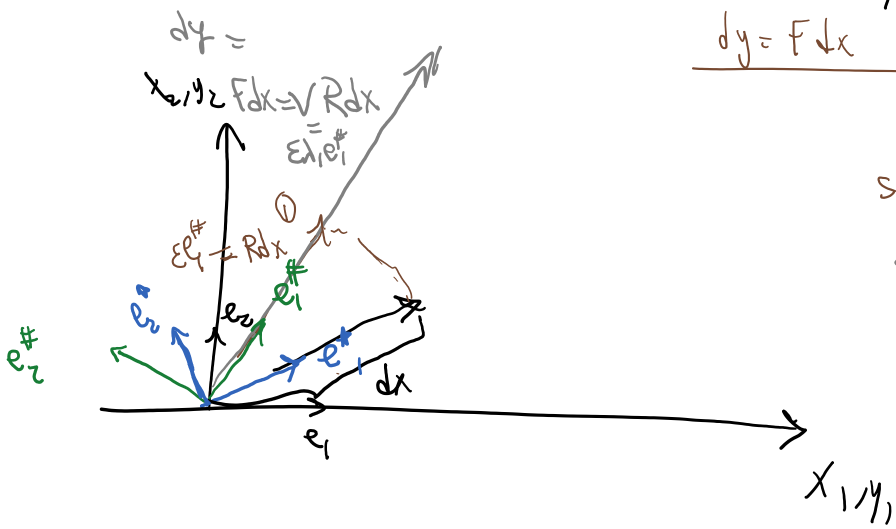
Right path   eigenvector

$$U dx = U \varepsilon e_1^* = \varepsilon \big(\underbrace{U e_1^*}_{\lambda_1 e_1^*}\big)$$

step 2 rotation

$$dy = F dx = R(U dx) = R(\varepsilon \lambda_1 e_1^*)$$

Left path



step 1 $R dx$ rotation

step 2   stretch

$$V R dx = V(\varepsilon e_1^\#) = \varepsilon V e_1^\#$$

eigenvector of V

Whether we go through the right or the left path

$$\boxed{\text{Eigenvectors of } U \text{ map to eigenvectors of } V \text{, stretch by eigenvalues } \lambda \text{ (of } U \& V)}$$

Expansion of C and other strain tensors

$$C = F^t F$$ right Cauchy Green deformation tensor

$$F = \nabla y_x$$

$$y = x + u$$ displacement

$$= \nabla (u + x)/x = \nabla u_x + I$$

$\underbrace{H}$ displacement gradient

$$F = H + I, \quad C = F^t F = (H+I)^t (H+I) = (H^t + I)(H+I) =$$

$$\underbrace{H^t H}_{\substack{\text{second order} \\ \text{in terms of } \frac{\partial u}{\partial x}}} + \underbrace{H^t + H}_{\text{first order terms}} + \underbrace{I}_{\text{0th order term}}$$

Def. 81

$dx_i \approx \frac{\partial y}{\partial x} dx$ $\quad$ dx mostly

$$G = \frac{1}{2}(C - I) = \frac{1}{2}(H^t H + H + H^t)$$

Green - St Venant strain

$$= \underbrace{\frac{1}{2}(H + H^t)}_{E} + \frac{1}{2} H^t H$$

$$G = E + \frac{1}{2} H^t H$$

$E = \frac{1}{2}(H + H^t)$ small deformation gradient (H) / infinitesimal strain tensor

Indicial notation formulas for C, G, and E

$$C_{ij} = (F^t F)_{ij} = (F^t)_{im} F_{mj} = F_{mi} F_{mj}$$

$$F_{mi} = \frac{\partial y_m}{\partial x_i} = \frac{\partial (u_m + x_m)}{\partial x_i} = \frac{\partial u_m}{\partial x_i} + \frac{\partial x_m}{\partial x_i} = H_{mi} + \delta_{mi}$$

$H = \nabla u_x$

$$F_{mj} = H_{mj} + \delta_{mj}$$

$$C_{ij} = (H_{mi} + \delta_{mi})(H_{mj} + \delta_{mj}) =$$

$$H_{mi} H_{mj} + \delta_{mi} H_{mj} + H_{mi} \delta_{mj} + \delta_{mi} \delta_{mj}$$

$$H_{mi} H_{mj} + \underbrace{\delta_{mi} H_{mj}}_{H_{ij}} + \underbrace{H_{mi}\delta_{mj}}_{H_{ji}} + \delta_{mi}\delta_{mj}$$

$$C_{ij} = \delta_{ij} + H_{ij} + H_{ji} + H_{mi} H_{mj}$$ in indicial expression of $C = (H+I)^t (H+I)$

$$G = \frac{C - I}{2} \rightarrow G_{ij} = \frac{1}{2}\left(\underbrace{H_{ij} + H_{ji}}_{E} + H_{mi} H_{mj}\right)$$

$$E = \frac{H + H^t}{2} \qquad E_{ij} = \frac{H_{ij} + H_{ji}}{2} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$$

Rigid body motion, $C, G, E = ?$

$$y = Qx + c \rightarrow F = \nabla y_x = Q$$

$$C = F^t F = Q^t Q = I$$

$\underbrace{U, V, C, B = I}_{\text{rigid motion}}$

$$G = \frac{C - I}{2} = \underline{\underline{0}}$$

this is a great strain measure (rigid motion $\rightarrow$ strain $= 0$)

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ in 3D

$$E = G - \frac{1}{2} H^t H = 0 - \frac{1}{2} H^t H$$

for a rigid motion $E \neq 0$

but it's close to zero ($O(\varepsilon^2)$ if $H = O(\varepsilon)$)