CM2021/10/19

Tuesday, October 19, 2021 4:31 PM

Large deformation gradient (no approximation)

$dy = F\,dy$

$F = R\,U = V\,R$

rotations

stretches

Infinitesimal theory (uses u rather than y)

$u = y - x$

$H = \nabla u_{/x} = \nabla y_{/x} - I = F - I$

$du = (E + W)\,dx$

$\text{strain} = \frac{H + H^T}{2}$

$W = \frac{H - H^T}{2}$ infinitesimal rotation

exact — infinitesimal

R ⟷ W

stretch

U ~ E strain

strain ~ stretch − 1

Rigid motion

R constant, not a function of x

$U = V = I$

exact rigid

$W \neq 0$ constant

$E = 0$

infinitesimal rigid motion

Summary of rigid motion

① $\forall x_1, x_2 \quad |x_1 - x_2| = |y_1 - y_2|$

$\Updownarrow$

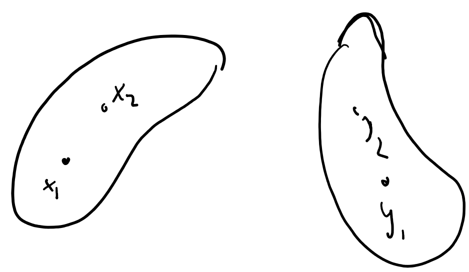
② $y(x) = Qx + c$

constant

$\Updownarrow$

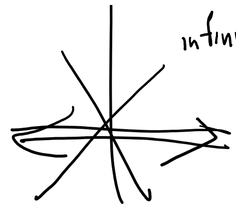
③ $C = I \qquad C = F^T F = Q^T Q = I$

$\Updownarrow$



infinitesimal rigid

④ $G = 0$ $\quad G = \frac{1}{2}(C - I)$



$\varepsilon = 0$

$E = O(\varepsilon^2)$

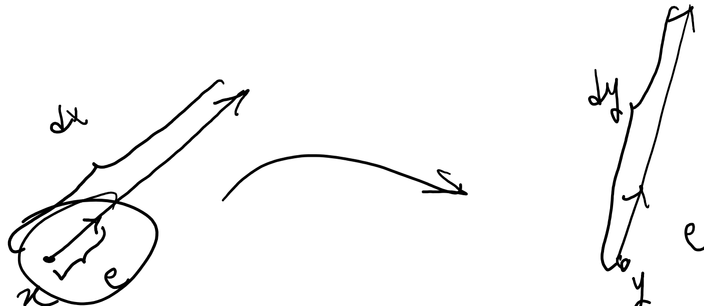
$G = O(\varepsilon^2)$

ε is a measure of H

Strain

1. Normal strain

$C = F^t F$
$U = \sqrt{C}$



$B = FF^t$
$V = \sqrt{B}$

Lagrangian (reference)

Eulerian / spatial / current configuration

The goal is to compute the change of length for a given direction:
- Lagrangian: e is given
- Eulerian: e* is given

Strain = normalized change of length

$$\frac{|dy| - |dx|}{|dx|} \quad \text{or} \quad \frac{|dy| - |dx|}{|dy|}$$

$|dy| = \sqrt{dx \cdot C dx} \qquad dx = |dx| e$

$$\varepsilon(x, e) = \frac{|dy| - |dx|}{|dx|} = \frac{\sqrt{(dx\, e) \cdot C (dx\, e)} - |dx|}{|dx|}$$

x: base point
e: orientation in ref. configuration

$= e \cdot Ce - 1$

does not depend on $|dx|$

definition makes sense for $|dx| \to 0$ though

④ $\varepsilon(x, \vec{e}) = \sqrt{e \cdot Ce} - 1$

(1) $\varepsilon(x, \vec{e}) = \sqrt{e.Ce} - 1$

x, e are in reference configuration

Lagrangian $(C, U), x$

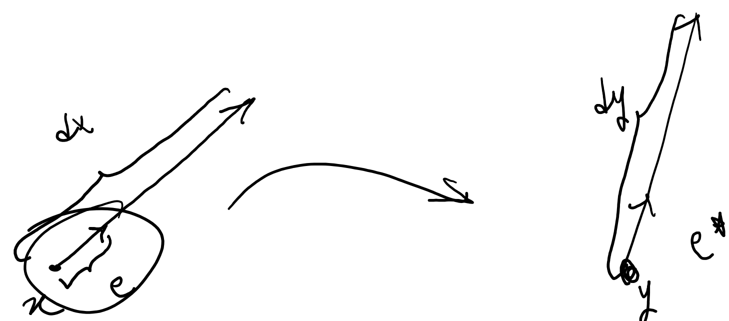
(2) $\varepsilon(y, e^*) = \sqrt{e^* B e^*} - 1$ current configuration

$B = FF^t$

Eulerian (B, V)

$C = F^tF$

$U = \sqrt{C}$



$B = FF^t$

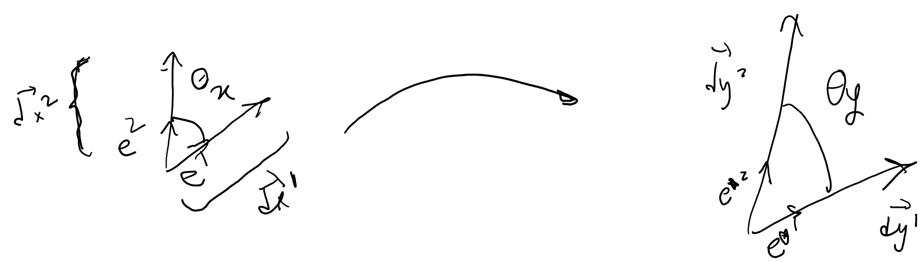
$V = \sqrt{B}$

Lagrangian (reference)

Eulerian / spatial / current configuration

(3)

B. Shear strain (change of angle between two directions)



e^1, e^2 given $\rightarrow \theta_y$ Lagrangian

$e^{*1}, e^{*2} \rightarrow \theta_x$ Eulerian

$$\cos\theta_y = \frac{\vec{dy}^1 \cdot \vec{dy}^2}{|\vec{dy}^1|\,|\vec{dy}^2|} \qquad dy^1 = F dx^1, \quad dy^2 = F dx^2$$

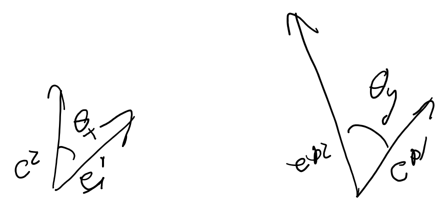
$$= \frac{F dx^1 \cdot F dx^2}{\sqrt{F dx^1 \cdot F dx^1}\sqrt{F dx^2 \cdot F dx^2}} = \frac{(|dx^1| e^1) \cdot C |dx^2| e^2}{\sqrt{|dx^1| e^1 \cdot C |dx^1| e^1}\sqrt{|dx^2| e^2 \cdot C |dx^2| e^2}} \Rightarrow$$

(II) $\cos\theta_y = \frac{e^1 \cdot Ce^2}{\quad}$

II $\quad \cos\theta_y = \dfrac{e^1\cdot Ce^2}{\sqrt{e^1\cdot Ce^1}\,\sqrt{e^2\cdot Ce^2}}$

Shear strain $\quad \theta_y - \theta_x$

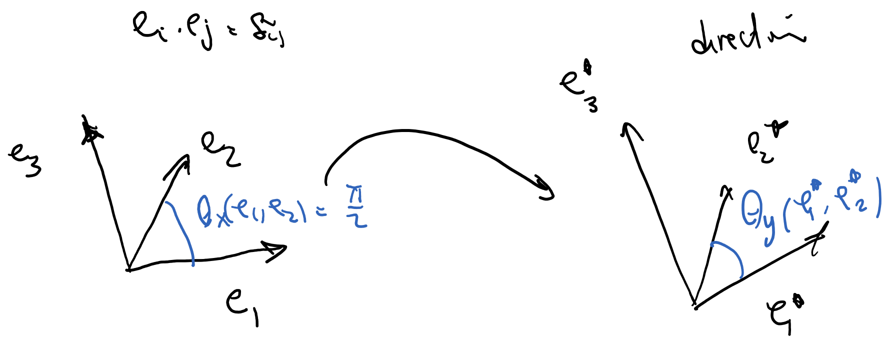
Lagrangian



Eulerian
$$\cos\theta_x = \frac{e^{\phi 1}\cdot Be^{\phi 2}}{\sqrt{e^{\phi 1}\cdot Be^{\phi 1}}\,\sqrt{e^{\phi 2}\cdot Be^{\phi 2}}}$$

Let's choose a coordinate system and define NORMAL and SHEAR strains for that coordinate system:

$e_i\cdot e_j = \delta_{ij}$



We have 3 normal strains: how much each direction e_i is changed in a nondimensional form:

I $\quad \longrightarrow \quad \varepsilon(x, e_i) = \sqrt{e_i\cdot C e_i} - 1$

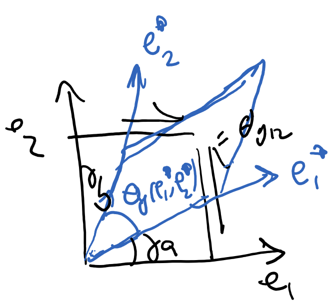
no summation

We have 3 shear strains, the change of angle between two axis directions

$\theta_x(e_i, e_j) = \frac{\pi}{2} \qquad i\neq j$

engineering shear strain $\quad \gamma_{ij} = \theta_x(e_i, e_j) - \theta_y(e_i^\phi, e_j^\phi)$

2D



$$\gamma_{12} = \underbrace{\frac{\pi}{2}}_{\theta_{x_{12}}} - \theta_{y_{12}} = \gamma_a + \gamma_b$$

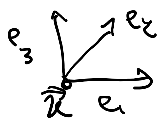
Useful identity $\quad \sin\gamma_{12} = \sin(\overset{\pi/2}{\widetilde{\theta_{x_{12}}}} - \theta_{y_{12}})$

$$= \sin\left(\frac{\pi}{2} - \theta_{y_{12}}\right)$$

$$= \cos \theta_{y_{12}} = \frac{e_1 \cdot C e_2}{\sqrt{e_1 \cdot C e_1}\,\sqrt{e_2 \cdot C e_2}}$$

from (II)

Summary of strains in a coordinate system



$$\underbrace{\varepsilon_{i\underline{i}}}_{\text{no summation}}(x) = \varepsilon(x, e_i) = \sqrt{e_i \cdot C e_{\underline{i}}} - 1$$

3 normal $\varepsilon_{i\underline{i}}$

$$\underbrace{\sin \gamma_{ij}}_{\text{change of angle between } e_i, e_j} = \frac{e_i \cdot C e_j}{\sqrt{e_i \cdot C e_{\underline{i}}}\,\sqrt{e_j \cdot C e_{\underline{j}}}}$$

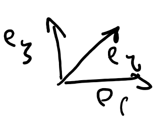
3 shear strains $\gamma_{ij} = \sin^{-1}(RHS)$

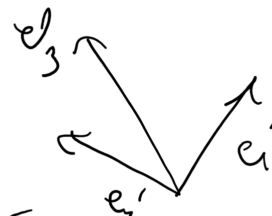
(III)

Can we define a strain tensor from this?

$$\underbrace{[\varepsilon]}_{\text{2 indexed array}} = \begin{bmatrix} \varepsilon_{11} & \gamma_{12} & \gamma_{13} \\ & \varepsilon_{22} & \gamma_{23} \\ \text{sym.} & & \varepsilon_{33} \end{bmatrix}$$

normal strains





do we have this property

$$[\varepsilon]' = \begin{bmatrix} \varepsilon_{11}' & & \\ & \ddots & \\ & & \end{bmatrix}$$

$$\varepsilon_{ij}' = Q_{im} Q_{jn} \varepsilon_{mn}$$

Coordinate transformation matrix

This does not hold! ε is not a tensor

Now that we know what normal and shear strains are, is there a path to define a "strain tensor" that encompasses both normal and shear strains?

We first start by discussing what infinitesimal deformation tensor is:

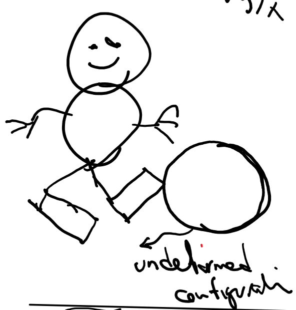
$y = f(x)$

$dy = F\,dx$

$F \rightarrow \nabla y_{/x}$ deformation gradient

$u = \vec{f}(x) - x$

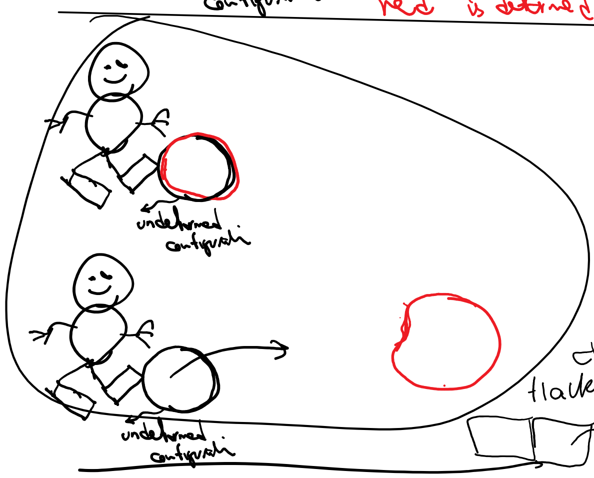
$H = \nabla u_{/x}$ → displacement gradient



red is deformed

small u?

small $H = \nabla u_{/x}$?



yes

yes

infinitesimal strain theory

strain $\varepsilon = \frac{H + H^t}{2}$

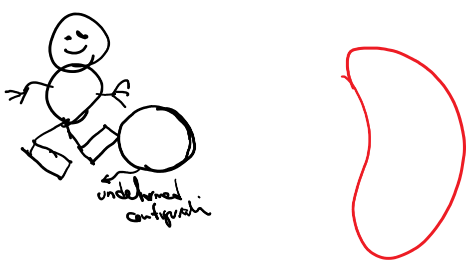
no

yes

challenge here is tracking contact points

$\omega = \frac{H - H^t}{2}$ rotation

eventually hits that wall



yes

(or H could be not)

No

difficult problem we need to use

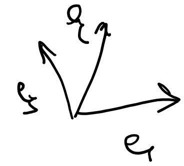
$F = RU = VR$

$C,\ B$

FINITE deformation

We want to find the approximate form of normal and shear strains under infinitesimal deformation theory:

$$\varepsilon(x, e_i) = \sqrt{C_{\underline{ii}}} - 1 \qquad C_{\underline{ii}} = e_i \cdot C e_{\underline{i}}$$



$$C = I + 2\overbrace{\left(\frac{H + H^t}{2}\right)}^{E} + H^t H$$

$$= I + 2E + O(\varepsilon^2)$$

H small

H or $\varepsilon \rightarrow$ small number $\ll 1$

$$C_{\underline{ii}} = 1 + \underbrace{2E_{\underline{ii}}}_{\text{order of } \varepsilon} + O(\varepsilon^2) \qquad \text{eq-i}$$

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \underbrace{\frac{1}{2}\left(\frac{1}{2}-1\right)x^2 + \cdots}_{O(x^2)} \qquad \text{eq-ii}$$

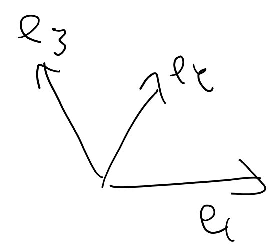
$x < 1$

$$\varepsilon(x, e_i) = \sqrt{C_{\underline{ii}}} - 1 = \qquad \text{eq iii}$$

$$= \sqrt{1 + \underbrace{2E_{\underline{ii}} + O(\varepsilon^2)}_{x \text{ in eq (2)}}} - 1 = \underbrace{1 + \frac{1}{2}\left(2E_{\underline{ii}} + O(\varepsilon^2) + \text{HOTs}\right)}_{\text{from eq-ii}} - 1$$

$$\boxed{\begin{gathered} \varepsilon(x, e_i) = E_{\underline{ii}} + O(\varepsilon^2) \quad \text{no summation on } i \\ \varepsilon = |H| \\ E = \frac{H + H^t}{2} = \frac{\nabla u + (\nabla u)^t}{2} \end{gathered}}$$

γ'



normal strains

$$E = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ & E_{22} & E_{23} \end{bmatrix} \rightarrow \text{shear strains}$$

for INFINITESIMAL

$$E = \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ & E_{22} & E_{23} \\ & & E_{33} \end{pmatrix} \longrightarrow$$

for INFINITESIMAL theory $|H| = \varepsilon \ll 1$