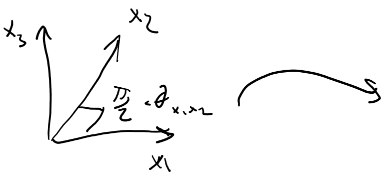
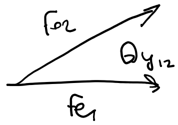
CM2021/10/21

Thursday, October 21, 2021 4:28 PM

Relation of off-diagonal values of E and shear strain:





$\gamma_{12} = \frac{\pi}{2} - \theta_{y_{12}}$

from last time

$$\sin \gamma_{ij} = \frac{e_i \cdot C e_j}{\sqrt{e_i \cdot C e_i}\,\sqrt{e_j \cdot C e_j}} \qquad i \neq j$$

$$C = I + 2\underbrace{\left(\frac{H + H^t}{2}\right)}_{E} + \underbrace{H^t H}_{O(\varepsilon^2)}, \ \varepsilon = |H| \qquad = C = I + 2E + O(\varepsilon^2)$$

$$e_i \cdot C e_j = e_i \cdot (I + 2E + O(\varepsilon^2)) \cdot e_j = \underset{0}{\delta_{ij}} + 2E_{ij} + O(\varepsilon^2)$$

for numerator $i \neq j$

denominator

$$e_i \cdot C e_i = \delta_{ii} + 2\underbrace{E_{ii}}_{O(\varepsilon)} + O(\varepsilon^2) = 1 + O(\varepsilon)$$

no summation on i

denominator

$$\sin \gamma_{ij} = \frac{2E_{ij} + O(\varepsilon^2)}{\sqrt{1 + O(\varepsilon)}\,\sqrt{1 + O(\varepsilon)}} = \frac{2E_{ij} + O(\varepsilon^2)}{(1 + \frac{1}{2}O(\varepsilon))(1 + \frac{1}{2}O(\varepsilon))}$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x + O(x^2)$$

$|x| \ll 1$

$$\sin \gamma_{ij} = 2\underbrace{E_{ij}}_{O(\varepsilon)} + O(\varepsilon^2)$$

small number

$$\boxed{\gamma_{ij} = 2E_{ij} + O(\varepsilon^2)}$$

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{5!} \ldots$$

as $x \to 0$

$\sin x \approx x$ $O(x^3)$ error

for infinitesimal deformation theory

tensorial strains

$$E_{ij} = \frac{\gamma_{ij}}{2}$$

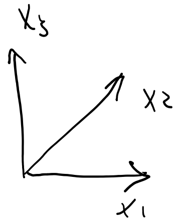
half of angle change

engineering strain $\gamma_{12} = 2E_{12}$

$$E = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ & E_{22} & E_{23} \\ \text{sym} & & E_{33} \end{bmatrix}$$

tensorial (half of engineering) shear strains between e_i & e_j ($i \neq j$)

normal strains along e_1, e_2, e_3

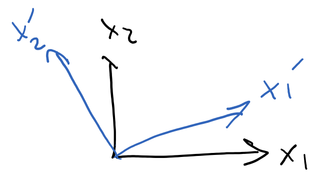


$$= \frac{\overbrace{\nabla u}^{H} + \nabla u^T}{2}$$

u is displacement field : tensor $\rightarrow$ ∇u is a tensor

$\rightarrow$ $E = \dfrac{\nabla u + \nabla u^T}{2}$ is a tensor & follows coordinate transformation rules

$$E'_{ij} = Q_{im} Q_{jn} E_{mn}$$



will be Mohr-circle for sym 2nd order tensors discussed next week.

Likewise $W = \dfrac{\nabla u - \nabla u^T}{2}$ is a rotation which is the infinitesimal theory counterpart to R

Voigt notation:

E is a sym. 2nd order tensor

2D $E = \begin{bmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{bmatrix}$ independent values

$$\gamma = \begin{bmatrix} E_{11} \\ E_{22} \\ 2E_{12} \end{bmatrix} \rightarrow \gamma_{12} \text{ eng strain}$$

3D $E = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix}$

$$\gamma = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{12} \\ 2E_{23} \\ 2E_{31} \end{bmatrix}$$

the order here can change

1-array obviously not a vector

For finite deformation theory where H is no small, we need to use F = RU or F = VR

- U and V encompass the physical deformation of material (stretching and change of angle)
- R is a rotation

U <-> C <-> G = 0.5 (C - 1)
V <-> B

---- a constitutive equation takes U and computes stress (and then incorporate the rotation part)

U is not a matrix of normal & shear strains

the question is what "strain" measure best is related to stress

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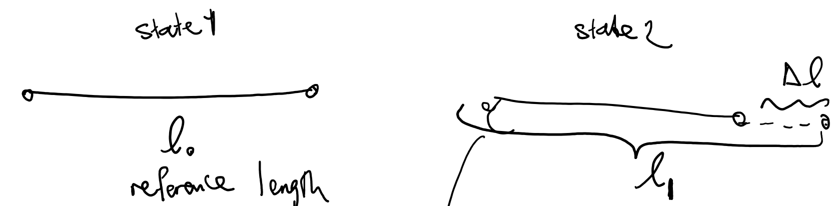
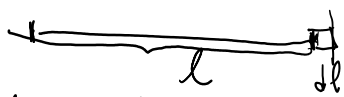
Background on Logarithmic strain



$$\varepsilon = \frac{\text{change of length}}{\text{reference length}} = \frac{\Delta l}{l_0} = \frac{l_1 - l_0}{l_0}$$

← reference length

our initial definition of strain



$$\varepsilon_{Lin} = \int_{l_0}^{l} d\varepsilon = \int_{l_0}^{l_1} \frac{dl}{l_0} = \frac{\Delta l}{l_0}$$

l_0: initial length

Logarithmic definition

$$\varepsilon_{Log} = \int d\varepsilon = \int_{l_0}^{l} \underbrace{\frac{\text{change of length}}{\text{current length}}}_{\frac{dl}{l}} = Ln\left(\frac{l_1}{l_0}\right)$$

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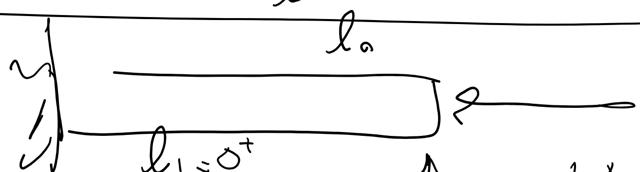
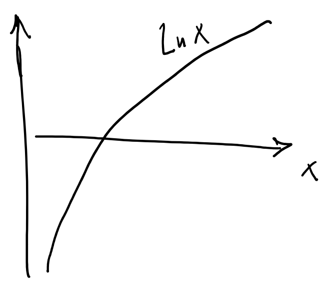
we push this all the way to $l_1 = 0^+$



$$\varepsilon_{Lin} = \frac{l_1 - l_0}{l_0} = \frac{0 - l_0}{l_0} = -1$$

$$\varepsilon_{Log} = Ln\left(\frac{l_1}{l_0}\right) = Ln\left(0^+\right) = -\infty$$



simple const. eqn

$$\underset{\text{stress}}{\sigma} = \underset{\text{constant}}{E}\, \varepsilon$$

To get the correct stress, we have two choices:

1. A simple linear const equation like above may be acceptable if we use logarithmic strain

2. Use linear strain BUT have more complex constitutive equation that gives -infinity stress for -1 linear strain.

In infinitesimal theory there is no distinctions between these and in fact as mentioned before E is perfect.

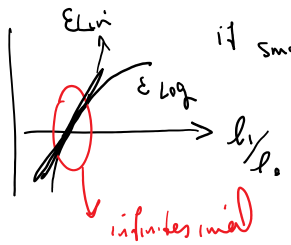
relation to here

$$\varepsilon_{Log} = Ln\left(\frac{l_1}{l_0}\right) = Ln\left(\frac{l_0 + \Delta l}{l_0}\right) = Ln\left(1 + \left(\frac{\Delta l}{l_0}\right)\right)$$

Lin $H = \frac{du}{dx}$ if small

$$= Ln(1 + \varepsilon_{Lin})$$

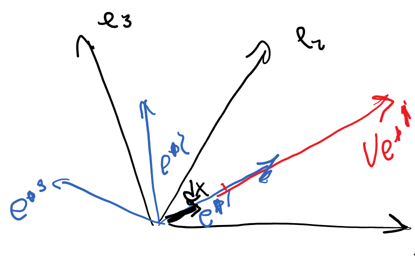
$$= \varepsilon_{Lin}^2 + O(\varepsilon_{Lin}^2)$$



infinitesimal theory they're the same

Background -> discussion of suitable strains for finite deformation theory

e^{*i} principal axes of C (& U)



$Ve^{*1} = \lambda_V^1 e^{*1}$

eigen value #1 of U

$dy = F\,dx$

if $dx = \varepsilon\, e^{*i}$

ε: small number

$$dy = RU\,dx = R(U\varepsilon e^{*i}) = \varepsilon R(Ue^{*i}) = \varepsilon R \lambda_U^i e^{*i}$$

$$= (\varepsilon \lambda_U^i)\, Re^{*i}$$

current length

$|dy| = \varepsilon \lambda^i$

$|dx| = \varepsilon$

initial length

$$\text{Linear strain} = \frac{|dy| - |dx|}{|dx|} = \frac{\varepsilon \lambda^i - \varepsilon}{\varepsilon} = \lambda_U^i - 1$$

for fiber along e^{*i}

$$\text{Log strain} = Ln\left(\frac{|dy|}{|dx|}\right) = Ln(\lambda_U^i)$$

in * system

$$\varepsilon_{Lin}(U) = \begin{bmatrix} \lambda_U^1 - 1 & 0 & 0 \\ 0 & \lambda_U^2 - 1 & 0 \\ 0 & 0 & \lambda_U^3 - 1 \end{bmatrix} = U - 1 \text{ expressed in *}$$

these are linear normal and shear strains and shear strains

$$\varepsilon_{Log}(U) = \begin{bmatrix} \ln(\lambda_U^1) & 0 & 0 \\ 0 & \ln(\lambda_U^2) & 0 \\ 0 & 0 & \ln(\lambda_U^3) \end{bmatrix} = \ln U$$

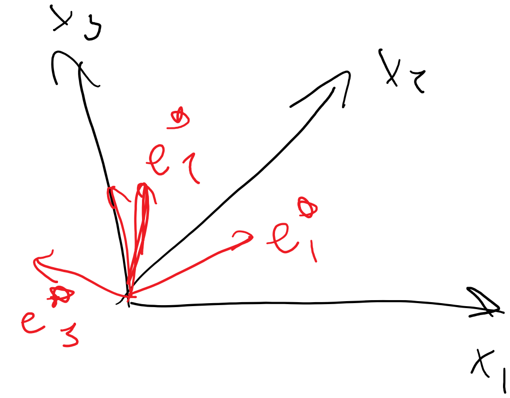
// in * system

logarithmic normal strain

zero shear strain

$$[U]^* = \begin{bmatrix} \lambda_U^1 & 0 & 0 \\ 0 & \lambda_U^2 & 0 \\ 0 & 0 & \lambda_U^3 \end{bmatrix}$$

* system



Recall

$A v_1 = \lambda_1 v_1$ → eigenvector

eigenval

$$A U = U \Lambda$$

for a diagonalizable matrix

$$A = U \Lambda U^{-1}$$

$$f(A) = U f(\Lambda) U^{-1}$$

if A is sym. → $U = Q^t$ $f(U) = Q^T f(U)^* Q$