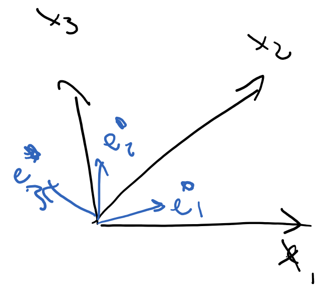
CM2021/10/26

Tuesday, October 26, 2021 4:33 PM

From last time



$$U - 1 = \begin{bmatrix} \lambda_{U_1}-1 & 0 & 0 \\ 0 & \lambda_{U_2}-1 & 0 \\ 0 & 0 & \lambda_{U_3}-1 \end{bmatrix}^{*}$$

$$\ln U = \begin{bmatrix} \ln \lambda_{U_1} & 0 & 0 \\ 0 & \ln(\lambda_{U_2}) & 0 \\ 0 & 0 & \ln(\lambda_{U_3}) \end{bmatrix}^{*}$$

are linear & logarithmic strains expressed in * system

we can define other strains

$U - \underline{1}$ linear strain

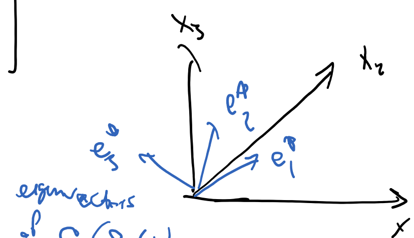
very simple no need to calculate U & its eigenvalues $G = \frac{1}{2}(F^tF - I)$

$G = \frac{1}{2}(U^2 - 1) = \frac{1}{2}(C - \underline{1})$ Green strain

$G^{(m)} = \frac{1}{m}(U^m - I)$ Generalized Green strain $m \neq 0$

$\ln U$ logarithmic strain

$$e(U) = \begin{bmatrix} e(\lambda_1^U) & 0 & 0 \\ 0 & e(\lambda_2^U) & 0 \\ 0 & 0 & e(\lambda_3^U) \end{bmatrix}^{*}$$ in the * system

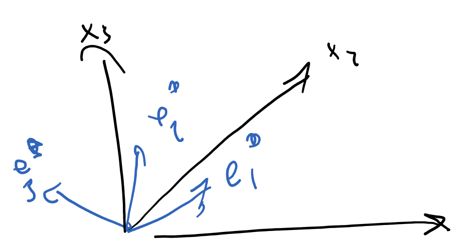


process of calculating e(U)

① Form $C = F^tF$

② Solve eigen vectors & values of C

e_1^*, e_2^*, e_3^* $\lambda_{C_1}, \lambda_{C_2}, \lambda_{C_3}$



in * system $[C]^* = \begin{bmatrix} \lambda_{C_1} > 0 & 0 & 0 \\ 0 & \lambda_{C_2} > 0 & 0 \\ 0 & 0 & \lambda_{C_3} > 0 \end{bmatrix}^{*}$

$Q = \begin{bmatrix} e_1^* \\ \hline e_2^* \\ \hline e_3^* \end{bmatrix}$

$[T]^* = Q[T]Q^t$

③ U in * system is $\left[\sqrt{\lambda_{C_1}} = \lambda_{U_1} \quad 0 \quad 0 \right.$

(3) U in * system is

$$[U]^* = \begin{bmatrix} \sqrt{\lambda_{C_1}} = \lambda_{U_1} & 0 & 0 \\ 0 & \sqrt{\lambda_{C_2}} = \lambda_{U_2} & 0 \\ 0 & 0 & \sqrt{\lambda_{C_3}} \end{bmatrix} = \lambda_{U_3}$$

$[T]^* = Q[T]Q^T$

$\Downarrow$

$[T] = Q^T[T]^*Q$

(4) e(U) is

$$[e(U)]^* = \begin{bmatrix} e(\lambda_{U_1}) & 0 & 0 \\ 0 & e(\lambda_{U_2}) & 0 \\ 0 & 0 & e(\lambda_{U_3}) \end{bmatrix}^*$$

(5) if needed

$$[e(U)] = Q^T \, [e(U)]^* Q$$

in x_1, x_2, x_3

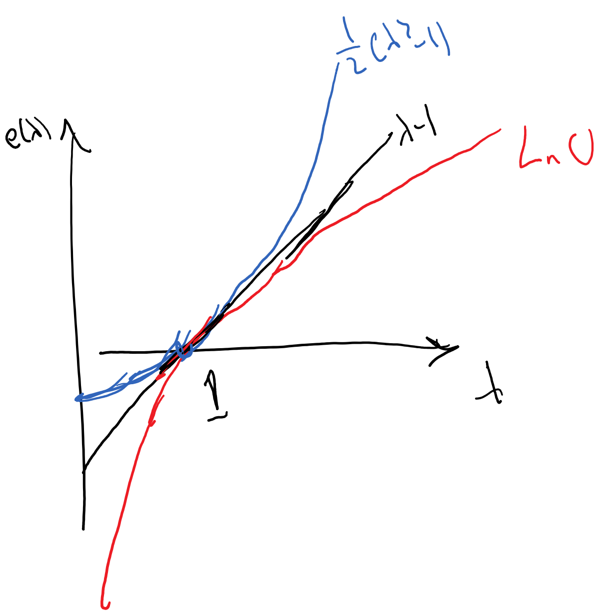
Are there any requirements on function e?

$e(\lambda) = \lambda - 1$ $\qquad$ $e(U) = U - I$ linear strain

$e(\lambda) = \frac{1}{2}(\lambda^2 - 1)$ $\qquad$ $e(U) = \frac{1}{2}(U^2 - 1) = \frac{1}{2}(C - I)$ Green-strain

$e(\lambda) = \frac{1}{m}(\lambda^m - 1)$ $\qquad$ $e(U) = \frac{1}{m}(U^m - I)$

$e(\lambda) = \operatorname{Ln}\lambda$ $\qquad$ $e(U) = \operatorname{Ln} U$



A) $e(1) = 0$

$e(\text{stretch} = 1) = 0$

strain: $|dy| = |dx|$ $= 0$

B) $e'(1) = 1$

to ensure difference of order $(\lambda-1)^2$ for different definitions

C) all are monolithically increasing

$e'(\lambda) > 0$ for all λ

stretch increases $\left(\frac{|dy|}{|dx|} \uparrow\right)$ so

$e(\lambda) = e(\lambda - 1 + 1)$

$= e(1) + e'(1)(\lambda-1) + \frac{1}{2}e''(1)(\lambda-1)^2 + HOT$

$e(\lambda) = e(\lambda - 1 + 1)$

$$= \underbrace{e(1)}_{0} + \underbrace{e'(1)}_{1}(\lambda-1) + \underbrace{\tfrac{1}{2}e''(1)(\lambda-1)^2 + \text{H.O.T}}_{O((\lambda-1)^2)}$$

Taylor's expansion

stretch increases ($\left|\frac{dy}{dx}\right| \uparrow$) so does strain

in HW 6 you'll prove

$$\underbrace{E_e}_{\text{strain for e function}} = e(U) = \overbrace{U - I}^{\text{exact linear strain}} + O(\varepsilon^2)$$

$$= \tfrac{1}{2}(H^T + H) + O(\varepsilon^2)$$

$$= E + O(\varepsilon^2)$$

any definition of finite strain matches $U - I$

and matches infinitesimal strain E

to within $O(\varepsilon^2)$ where $\varepsilon = \|H\|$

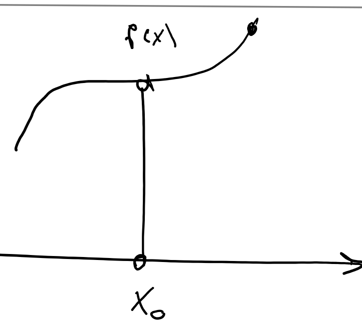
Big O notation

Example

$$f(x_0+\Delta x) = \underbrace{f(x_0) + \Delta x f'(x_0) + \tfrac{1}{2}\Delta x^2 f''(x_0) + \dots + \tfrac{1}{m!}\Delta x^m f^{(m)}(x_0)}_{}$$

Taylor's expansion

$g(x)$



I write this as

$$f(x_0+\Delta x) = f(x_0) + \Delta x f'(x_0) + O(\Delta x^2) \quad \text{as } \Delta x \to 0$$

we claim $g(x) = \underbrace{\tfrac{1}{2}\Delta x^2 f''(x_0) + \dots + \tfrac{1}{m!}\Delta x^m f^{(m)}(x_0) \dots}$

is $O(\Delta x^2)$

$$|g(x)| = \underbrace{|\Delta x|^2 \left\| \tfrac{1}{2} f''(x_0) + \tfrac{1}{6}\Delta x f'''(x_0) \dots \right\|}$$

can we claim

there exist a C such that as $\underline{\Delta x \to 0}$

$$|g(x)| \le C\,|\Delta x|^2$$

here $C = |f''(x_c)|$ would do it

> **Def.** we call $g(x, \Delta x) = O(\Delta x^p)$ if
> for $\Delta x \to 0$ there exists a C such that
> $$|g(x, \Delta x)| \le C\,|\Delta x|^p$$

Recall some norm of H

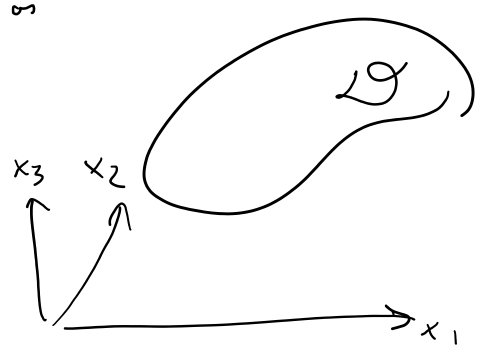
$$\varepsilon = \|H\|$$

in TAM 551 $\|\cdot H\|$ is defined as

$$\|H\| = \max_{\substack{x_1, x_2, x_3 \\ 1 \le i, j \le 3}} H_{ij}(x) \quad (\star)$$

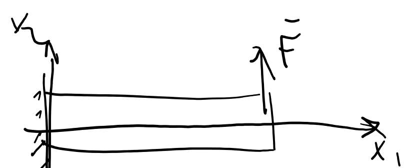
$\forall x \in \mathcal{D}$

in a given coordinate system



HW
Example

$$y_1 = x_1 + \alpha(1 - x_1)x_2$$
$$y_2 = \left[1 - \frac{\alpha^2}{2}(1 - x_1)^2\right] x_2$$



approx. deformation for this

$$F = \nabla y_{/x} = \begin{bmatrix} 1 - \alpha x_2 & \alpha(1 - x_1) \\ \alpha^2 (1 - x_1)(x_2) & 1 - \frac{\alpha^2}{2}(1 - x_1)^2 \end{bmatrix}$$

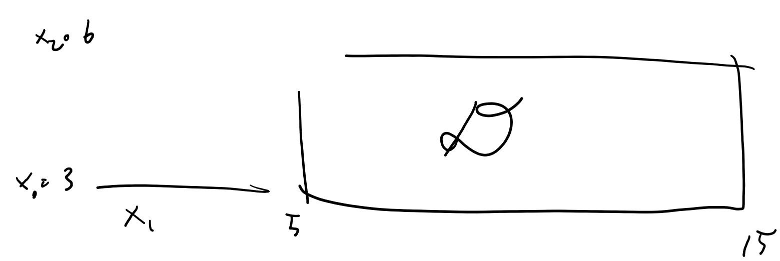
$$H = F - I = \alpha \begin{bmatrix} -x_2 & (1 - x_1) \\ \alpha(1 - x_1)x_2 & -\frac{\alpha}{2}(1 - x_1)^2 \end{bmatrix}$$

if $\alpha \ll 1$

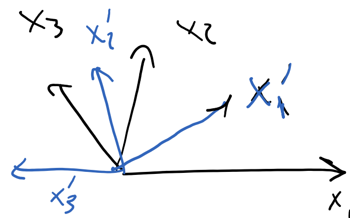
$|H_{ij}(x_1, x_2)| \ll 1$

for all x_1, x_2

we're in infinitesimal theory



the benefit of (*) is that it can easily be evaluated.
Disadvantage: It's value is coordinate-dependent



$$H'_{ij} = Q_{im} Q_{jn} H_{mn}$$

$$|H'_{ij}| \le \sum_{m=1}^{3} \sum_{n=1}^{3} |Q_{im}| |Q_{jn}| \underbrace{|H_{mn}|}_{\|H\|_{x_1,x_2,x_3}}$$

triangle inequality

$$\le \sum_{m=1}^{3} \sum_{n=1}^{3} 1 \times 1 \times \|H\|_{x_1,x_2,x_3}$$

Q orthonormal
$|Q_{im}| \le 1$

$$\frac{1}{9} \|H\|_{x_1,x_2,x_3} \le \|H\|_{x_1',x_2',x_3'} \le 9 \|H\|_{x_1,x_2,x_3}$$

basically if $\|H\|$ is small ($\ll 1$) in one coordinate system, it's small in all others $\rightarrow$ it suffices for checking infinitesimal theory condition

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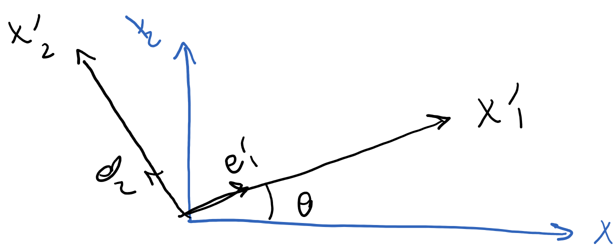
Mohr circle: coordinate transformation and eigensolution

Coordinate transformation of symmetric 2nd order tensors follows the Mohr circle rotations
3D is more complicated and is discussed in course notes

2D

$$Q = \begin{bmatrix} e_1' \\ e_2' \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$



$c = \cos\theta$, $s = \sin\theta$

Q rotates $(\ ) \leftrightarrow (\ )'$

(ST) $\phi' = \phi$
(VT) $v_i' = Q_{ij} v_j$ $\quad [v]' = Q[v]$
(TT) $T_{ij}' = Q_{im} Q_{jn} T_{mn}$ $\quad [T]' = Q[T]Q^T$
$\vdots$

$$E = \begin{bmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{bmatrix} \text{ is given}$$

because of sym ($E_{21} = E_{12}$)

we want to compute $E' = \begin{bmatrix} E_{11}' & E_{12}' \\ & E_{22}' \end{bmatrix}$

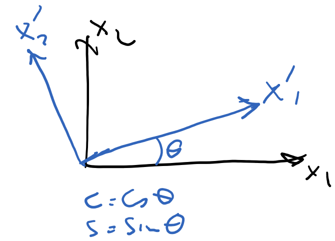
(TT)

$$[E]' = Q[E]Q^T$$

$$\rightarrow \begin{bmatrix} E_{11}' & E_{12}' \end{bmatrix} = \begin{bmatrix} c & s \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} \end{bmatrix} \begin{bmatrix} c & -s \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} E_{11}' & E_{12}' \\ E_{12}' & E_{22}' \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

$$E_{11}' = c^2 E_{11} + s^2 E_{22} + 2cs\, E_{12}$$
$$E_{22}' = s^2 E_{11} + c^2 E_{22} - 2cs\, E_{12}$$
$$E_{12}' = -cs(E_{11} - E_{22}) + (c^2 - s^2) E_{12}$$



(MA)

noting that $C = \cos(2\theta) = c^2 - s^2$

$S = \sin(2\theta) = 2cs$

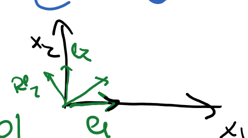
& using $c^2 = \frac{1+\cos 2\theta}{2}$

$s^2 = \frac{1-\cos 2\theta}{2}$

$$\begin{bmatrix} E_{11}' \\ E_{12}' \end{bmatrix} = \begin{bmatrix} \frac{E_{11}+E_{22}}{2} \\ 0 \end{bmatrix} + \overbrace{\begin{bmatrix} C & S \\ -S & C \end{bmatrix}}^{\text{rotation by } -2\theta} \begin{bmatrix} \frac{E_{11}-E_{22}}{2} \\ E_{12} \end{bmatrix}$$

Recall $R(\varphi)$

$= \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$

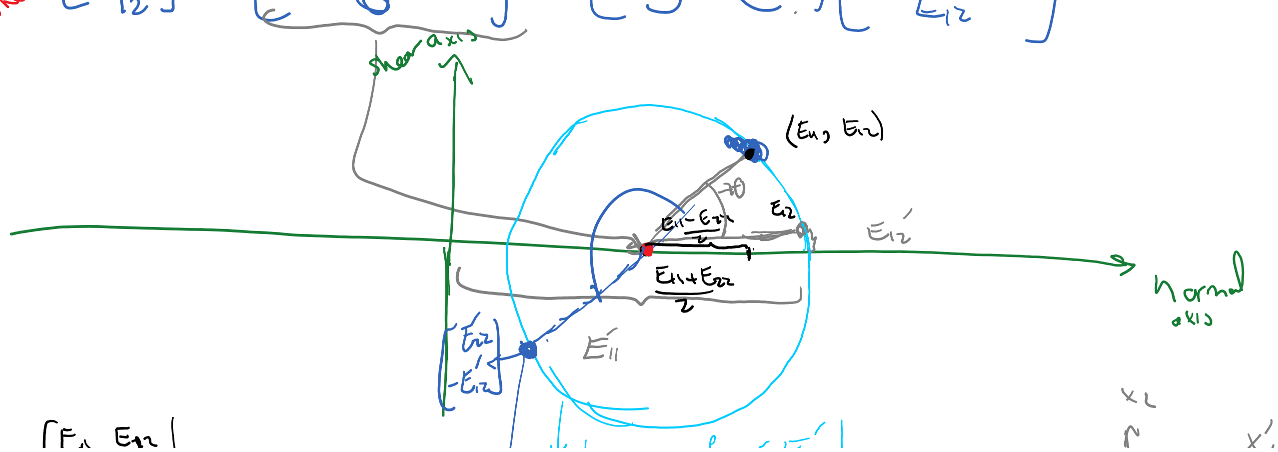


$R(e_1) = \text{Col}_1 \text{ of } R = \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}$

$R(e_2) = (\text{''})_2 = R = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

$\overline{2}$

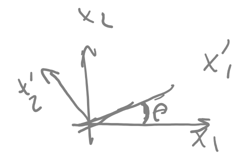
$$\begin{array}{l} \text{normal} \\ \text{shear} \end{array} \begin{bmatrix} E_{11}' \\ E_{12}' \end{bmatrix} = \begin{bmatrix} \frac{E_{11}+E_{22}}{2} \\ 0 \end{bmatrix} + \overbrace{\begin{bmatrix} C & S \\ -S & C \end{bmatrix}}^{\text{rotation by } -2\theta} \begin{bmatrix} \frac{E_{11}-E_{22}}{2} \\ E_{12} \end{bmatrix}$$



$[E_{11}\ E_{12}]$

$$\begin{bmatrix} E_{11} & E_{12} \\ & E_{22} \end{bmatrix}$$
given

locus of $\begin{bmatrix} E'_{11} \\ E'_{12} \end{bmatrix}$ for all other θ's



x'_1 $E_{x'_1x'_1} = E_{22}$

$E_{x'_1x'_2} = -E_{12}$