CM2021/11/02

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Relating Largangian (x fixed) and Eulerian (y fixed) rates

$$\frac{D}{Dt}\Big|_{x\text{-fixed}} \qquad \frac{\partial}{\partial t}\Big|_{y\text{-fixed}}$$

$$\underbrace{\frac{\partial \hat{T}(y,t)}{\partial t}\Big|_{y\text{-fixed}}}_{\text{Eulerian}} \overset{?}{=} \frac{D\,T(x,t)}{Dt}$$

TAM 551 "^" means some thing is expressed in y (Eulerian) rather than x

$$\frac{D\hat{T}(y,t)}{Dt}\Big|_{x\text{-fixed}} = \frac{\partial \hat{T}(y,t)}{\partial t} + \frac{\partial \hat{T}(y,t)}{\partial y_i}\underbrace{\boxed{\frac{\partial y_i}{\partial t}\Big|_{x\text{-fixed}}}}_{V_i \text{ velocity}}$$

chain rule

$$\frac{D\hat{T}(y,t)}{Dt} = \frac{\partial \hat{T}(y,t)}{\partial t} + (\nabla_y \hat{T})_i\, V_i \implies$$

for any tensor order

$$\boxed{\underbrace{\frac{DT}{Dt}\Big|_{x\text{-fixed}}}_{\text{Lagrangian (Material) rate}} = \underbrace{\frac{\partial \hat{T}}{\partial t}\Big|_{y\text{-fixed}}}_{\text{Eulerian (spatial) rate}} + \underbrace{\nabla_y T}_{\text{grad } T} \cdot V} \quad \text{(I)}$$

Example $T = V$ velocity

$$\frac{DV}{Dt}\Big|_{x\text{-fixed}} = \frac{\partial V}{\partial t} + \underbrace{\nabla_y \hat{V}}_{L} \cdot V$$

L

(2) $$\boxed{a = \frac{\partial \hat{v}(y,t)}{\partial t} + L \cdot \hat{V}(y,t) \quad L = \nabla_y V}$$

Lagrangian (x) Eulerian (y)

$\frac{D}{Dt} \longleftrightarrow \frac{\partial}{\partial t}$

$\text{Grad}\, T(x,t)\Big|_{x\text{-fixed}}$ $\quad$ $\text{grad}\, \hat{T}(y,t)\Big|_{y\text{-fixed}}$

$\nabla_x T$ $\quad$ $\nabla_y \hat{T}$

$\text{Div}\, T(x,t)$ $\quad$ $\text{div}\,(\hat{f}(y,t))$ $\quad$ HW

$= \text{trace}(\text{Grad}\, T)$ $\quad$ $= \text{trace}(\text{grad}\, \hat{T})$

Consider vector field w: $w(x,t)$, $\hat{w}(y,t)$

$$(\text{Grad}\, \hat{w})_{ij} = \frac{\partial \hat{w}_i(y,t)}{\partial x_j}\Big|_{x\text{-fixed}} = \frac{\partial \hat{w}_i(y,t)}{\partial y_k} \boxed{\frac{\partial y_k}{\partial x_j}}\Big|_{x\text{-fixed}}$$

$$= (\text{grad}\, \hat{w})_{ik}\, F_{kj}$$

(3) $$\boxed{\text{Grad}\, \hat{w} = \text{grad}\, \hat{w} \cdot F \iff \text{grad}\, \hat{w} = (\text{Grad}\, \hat{w})\, F^{-1}}$$

This holds true for any tensor order

$$(\text{Grad}\, \hat{T}(y,t))_{i_1 \ldots i_m j} = \frac{\partial \hat{T}_{i_1 \ldots i_m}}{\partial x_j} = \overbrace{\frac{\partial \hat{T}_{i_1 \ldots i_m}}{\partial y_k}}^{\text{grad}\, \hat{T}_{i_1 \ldots i_m k}} \overbrace{\frac{\partial y_k}{\partial x_j}}^{F_{kj}}$$

Another useful relationship is for J:

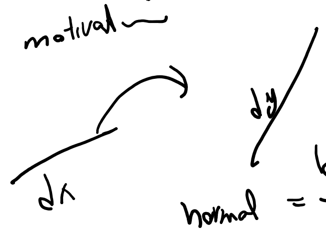
$$J = \det F \qquad\qquad F = \nabla y_{,x}$$

$$\frac{DJ}{Dt} = ?$$

material rate of J

side note $\quad J = V_{y/x} \longrightarrow J = \frac{dV_y}{dV_x} = \frac{\text{new volume}}{\text{old volume}}$

motivation



normal strain $= \frac{|dy| - |dx|}{|dx|}$

$$\varepsilon_v = J - 1 = \frac{\text{new vol} - \text{old vol}}{\text{old vol}}$$

$$= \underbrace{E_{11} + E_{22} + E_{33}}_{\text{trace}(E)} + O(\varepsilon^2) \qquad \varepsilon = \|H\|$$

HW

back to identity for $\frac{DJ}{Dt}$ material rate of J

$J = \det F$

$$\frac{D \det F}{Dt}$$

identity $\quad \frac{d \det A}{d\alpha} = \text{trace}\left(\frac{dA}{d\alpha} A^{-1} \right) \det A$

much simpler to calculate than

$A \to F$

$\alpha \to t$

$$\frac{D \det F}{Dt} = \text{trace}\left(\frac{DF}{Dt} F^{-1} \right) \underbrace{\det F}_{J} \qquad (a)$$

$$\left(\frac{DF}{Dt}\right)_{ij} = \frac{\partial}{\partial t}\left(\frac{\partial y_i}{\partial x_j} \right)\Big|_{x\text{-fixed}} = \frac{\partial}{\partial x_j}\underbrace{\left(\frac{\partial}{\partial t} y_i \right)\Big|_{x\text{-fixed}}}$$

$$= \frac{\partial^2 f}{\partial z \partial x} \qquad \frac{\partial^2 f}{\partial x \partial z}$$

$$= \frac{\partial}{\partial x_j} v_i \Big|_{x\text{-fixed}} = (\text{Grad } v)_{ij} \qquad \cancel{(\text{grad } v)_{ij}} \; y\text{-fixed}$$

$$\left(\frac{DF}{Dt}\right)_{ij} = (\text{Grad } v)_{ij} \quad \to \text{plug in } (c)$$

$$\frac{DJ}{Dt} = \frac{D \det F}{Dt} = \cancel{\text{trace}}\underbrace{\left(\text{Grad } v \cdot F^{-1}\right)} J$$

recall

$\underbrace{\text{Grad}\, T}_{\text{Lagrangian}} = \underbrace{\text{grad}\, T}_{\text{Eulerian}} \cdot F$

$(\ )\cdot F^{-1}$

$\text{grad } T = \text{Grad } T \cdot F^{-1}$

for $T = V$

(3)

$$\boxed{\frac{DJ}{Dt} = \text{trace}\underbrace{\left(\text{grad } v\right)}_{L} J = \text{trace}(L)\, J}$$

$L = \text{grad } v$ is often used in fluid mechanics

Summary of eqns 1 to 3

Lagrangian

$$v(X,t) = \frac{Dy(X,t)}{Dt} = \frac{Du(X,t)}{Dt}$$

Eulerian

$\hat{v}(y,t)$

$$\frac{DT}{Dt} = \frac{\partial \hat{T}(y,t)}{\partial t} + \text{grad } \hat{T}(y,t) \cdot \hat{v}$$

$$\nabla_X T = \text{Grad } T = \underbrace{\text{grad } \hat{T}(y,t)}_{\nabla_y \hat{T}} F$$

$$\frac{DJ}{Dt} = \text{trace}(\text{grad } v)\, F = \underbrace{\left(\frac{\partial v_1}{\partial y_1} + \frac{\partial v_2}{\partial y_2} + \frac{\partial v_3}{\partial y_3}\right)}_{\text{div } v} F$$

We'll see that div v = 0 corresponds to incompressibility condition

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Balance laws:

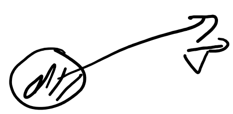
Newton's first law

$p = m\vec{v}$

linear momentum

$$\text{rate}(p) = \sum F$$

$$= \frac{d\,mv}{dt} = m\frac{dv}{dt} = ma$$



m

How do we take this to continuum

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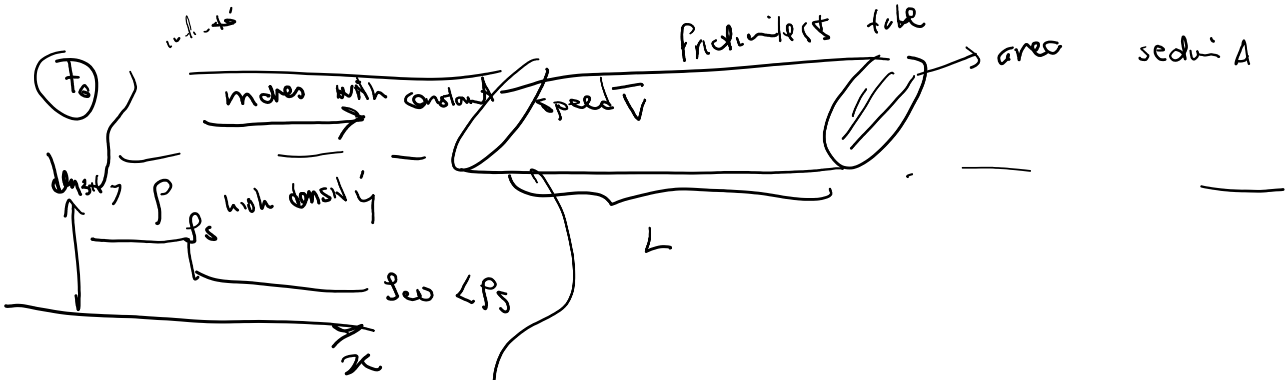
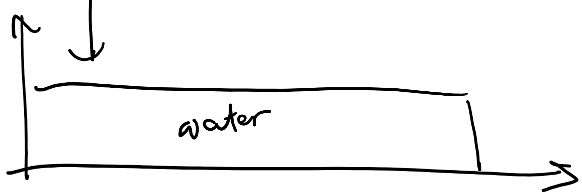


t_0



Linear momentum

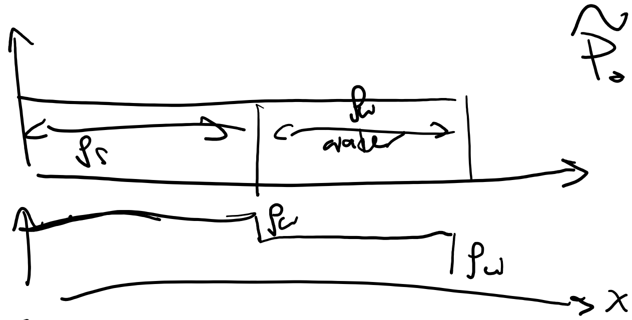
$= \text{Mass} \times \bar{V}$

$= \underbrace{(\underbrace{LA}_{\text{volume}} \cdot \rho_w)}_{\text{mass}} \bar{V}$

$\tilde{P}_0 = LA\rho_w \bar{V}$

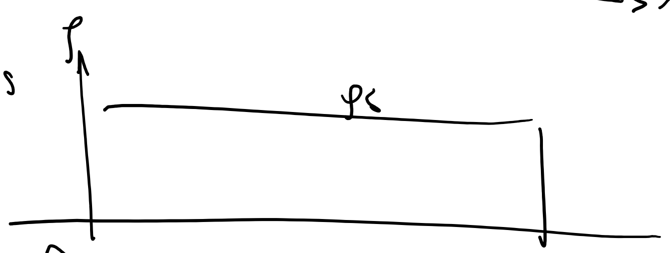
at a later time:

t_1



$\tilde{P}_1 = LA\left(\frac{\rho_w+\rho_s}{2}\right)\bar{V}$

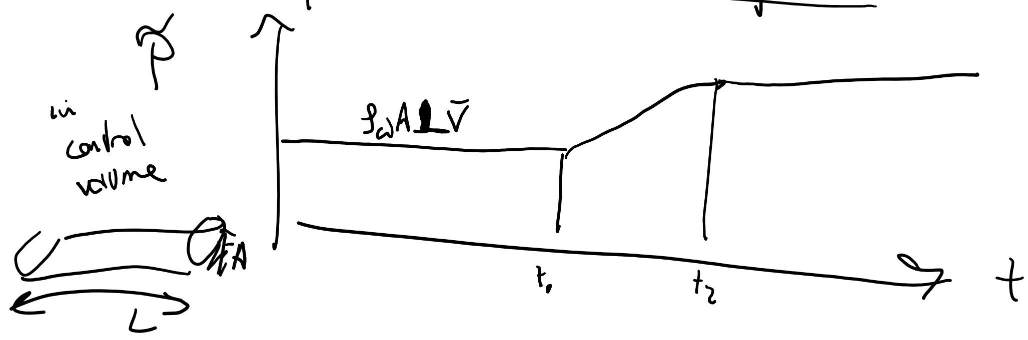
t_2 all fluids



time t_2 Mass $\times$ velocity

$\tilde{P}_2 = LA\rho_s\bar{V}$

$\tilde{P}$ in control volume



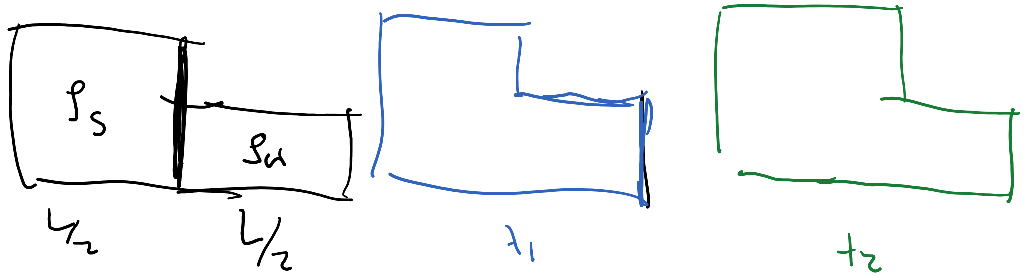
If way say $F = \text{rate}(\text{Mass} \times \text{velocity}) = \frac{d}{dt}\tilde{P}$

It says that for $t = t_0$ to t_2 $F \neq 0$

All the argument above is WRONG because we don't follow a blub of material in time (we are taking Eulerian time rate rather than Lagrangian time rate which is not physical)

what we should have done was to follow a blub of MATERIAL in time

consider this blub



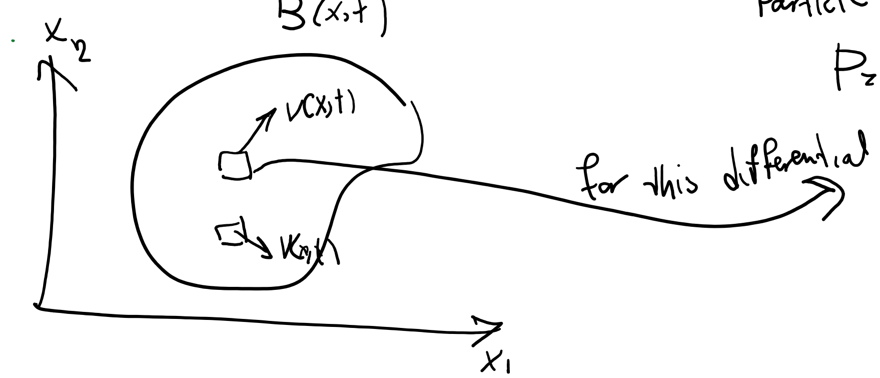
t_0

$$P = \underbrace{\frac{L(\rho_s + \rho_w)}{2} A}_{\text{Mass}} \overline{V} \qquad P = \qquad P_2 =$$

$$\frac{DP}{Dt} = F = 0$$

constant speed body does not experience net force

Now we know that we should take the material time derivative for balance of linear momentum. But how do we calculate linear momentum to begin with?



Particle

$P = mV$

$dP = (dm)\, V$ ← velocity

$= (\rho\, dV)$

mass density

Volume differential

dP

mass density differentiating

$$dP = \underbrace{(\rho \vec{v})}_{p} \, d\mathcal{V}$$

$$\boxed{P_B = \int_{B(t)} p \, d\mathcal{V}_x}$$

linear momentum for body B

linear momentum density = (" " per unit volume)

$p = \rho \vec{v}$

$$\sum F = \frac{D}{Dt} P_B$$

but calculating this material rate is a bit challenging.

Before that, let's note

$$P_B = \int_B p \, d\mathcal{V}$$

linear momentum

density of

$$M_B = \int_B \rho \, d\mathcal{V}$$

mass

density of mass (or as we call it "mass density")

in general

$$\Omega_B = \int_B \omega \, dV$$

volumetric density of Ω

example

linear momentum $\Omega = P$ $\qquad \omega = p = \rho \vec{V}$

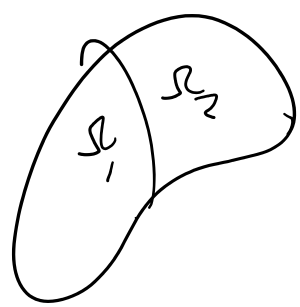
Mass $\Omega = M$ $\qquad \omega = \rho$

Energy $\Omega = \mathcal{E}$ $\qquad \omega = e_V$ volumetric energy density

1.8 Extensive Properties and their Densities.

In the previous sections we considered physical properties such as temperature that were associated with individual particles of the body. Certain other physical properties in continuum physics (such as for example mass, energy and entropy) are associated with parts of the body and not with individual particles.

Consider an arbitrary part $\mathcal{P}$ of a body $\mathcal{B}$ that undergoes a motion χ. As usual, the regions of space occupied by $\mathcal{P}$ and $\mathcal{B}$ at time t during this motion are denoted by $\chi(\mathcal{P}, t)$ and $\chi(\mathcal{B}, t)$ respectively, and the location of the particle p is $\mathbf{y} = \chi(p, t)$.



We say that Ω is an *extensive physical property* of the body if there is a function $\Omega(\cdot, t; \chi)$ defined on the set of all parts $\mathcal{P}$ of $\mathcal{B}$ which is such that

(i)

$$\Omega(\mathcal{P}_1 \cup \mathcal{P}_2, t; \chi) = \Omega(\mathcal{P}_1, t; \chi) + \Omega(\mathcal{P}_2, t; \chi) \qquad (1.30)$$

for all arbitrary disjoint parts $\mathcal{P}_1$ and $\mathcal{P}_2$ (which simply states that the value of the property Ω associated with two disjoint parts is the sum of the individual values for each of those parts), and

(ii)

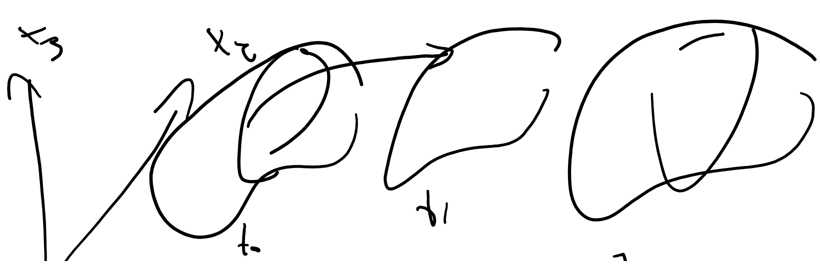
$$\Omega(\mathcal{P}, t; \chi) \to 0 \quad \text{as the volume of } \chi(\mathcal{P}, t) \to 0. \qquad (1.31)$$

Under these circumstance there exists a *density* $\omega(p, t; \chi)$ such that

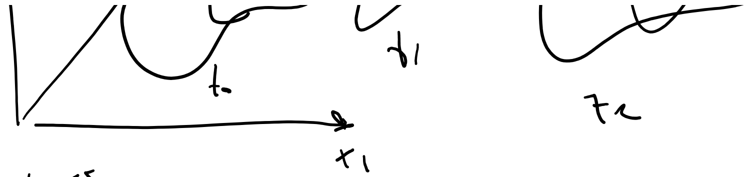
$$\Omega(\mathcal{P}, t; \chi) = \int_{\mathcal{P}} \omega(p, t; \chi) \, dp. \qquad (1.32)$$

In balance laws we need to compute the material rate of such integrals.
Examples:

$$M(t) = \int_{B(t)} \rho \, dv$$



$B(t)$



$$\frac{DM}{Dt} = 0$$ balance of mass

$$\frac{D\bar{P}}{Dt} = \sum F$$

linear momentum

$$P(t) = \int_{B(t)} p \, dV \qquad p = \rho \vec{v}$$

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the real challenge is calculating the rate of an integral whose integrand & domain of integration change.

$$\frac{D}{Dt}\Omega = \frac{D}{Dt}\int_{B_y} \omega(y,t)\, dV_y$$

changes by time

changes