CM2021/11/04

Thursday, November 4, 2021 2:49 PM

How do we calculate the rate of integrals whose domain and integrand are both time-dependent?

1D version

$$\frac{D}{Dt}\int_{y_A}^{y_B} g(y,t)\,dy = \int_{y_A}^{y_B} \frac{Dg}{Dt}(y,t)\,dy$$

simple when y_A, y_B are fixed

How about the case that y_A, y_B change

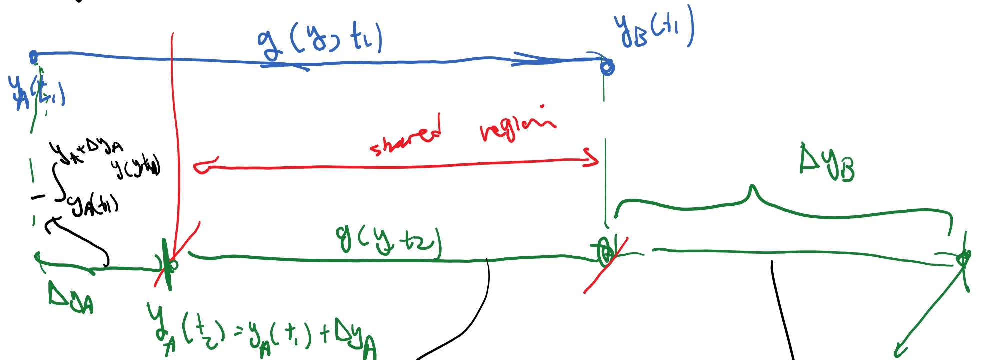
$$G = \int_{y_A(t)}^{y_B(t)} g(y,t)\,dy$$

@ $t = t_1$

$$G(t_1) = \int_{y_A(t_1)}^{y_B(t_1)} g(y,t_1)\,dy$$

@ $t = t_2$

$$G(t_2) = \int_{y_A(t_2)}^{y_B(t_2)} g(y,t_2)\,dy$$



$\Delta G = G(t_2) - G(t_1)$ = 3 contributions $\quad \int g(y,t_2) - g(y,t_1) \quad \int_{y_B}^{y_B+\Delta y_B} g(y,t_2)$

$$\frac{dG}{dt} = \lim \frac{\Delta G}{\Delta t} = \int_{y_A(t)}^{y_B(t)} \frac{dg}{dt}(y,t)\,dy - \underbrace{\frac{dy_A}{dt}}_{v_A} g(y_A) + \underbrace{\frac{dy_B}{dt}}_{v_B} g(y_B)$$

$$= \int_{y_A(t)}^{y_B(t)} \left.\frac{\partial g}{\partial t}\right|_{\substack{\text{Eulerian} \\ y\text{-fixed}}} dy + n_A v_A\, g(y_A) + n_B v_B\, g(y_B) \qquad \text{(I)}$$

we want to generalize this to 2D & 3D

TAM 551 2.6 The Transport & Localization Theorems

Theorem 145 (Transport Theorem) *Let $g \in C^1(\Im, \Re)$ be a spatial scalar field. Then*

$$\frac{d}{dt}\int_{\mathcal{P}_t} g(\mathbf{y},t)\,dV_y = \int_{\mathcal{P}_t}\left[\frac{\partial g}{\partial t}(\mathbf{y},t) + g_{,i}(\mathbf{y},t)\hat{v}_i(\mathbf{y},t) + g(\mathbf{y},t)\hat{v}_{i,i}(\mathbf{y},t)\right]dV_y \qquad L1$$

$$= \int_{\mathcal{P}_t}\left\{\frac{\partial g}{\partial t}(\mathbf{y},t) + [g\hat{v}_i]_{,i}(\mathbf{y},t)\right\}dV_y \qquad L2$$

$$= \int_{\mathcal{P}_t}\frac{\partial g}{\partial t}(\mathbf{y},t)\,dV_y + \int_{\partial\mathcal{P}_t} g(\mathbf{y},t)\left[\hat{\mathbf{v}}(\mathbf{y},t)\cdot\mathbf{n}(\mathbf{y},t)\right]dA_y, \qquad L3$$

where $\mathbf{n}(\mathbf{y},t)$ is the outward unit normal to $\partial\mathcal{P}_t$ at $\mathbf{y}$.[20]

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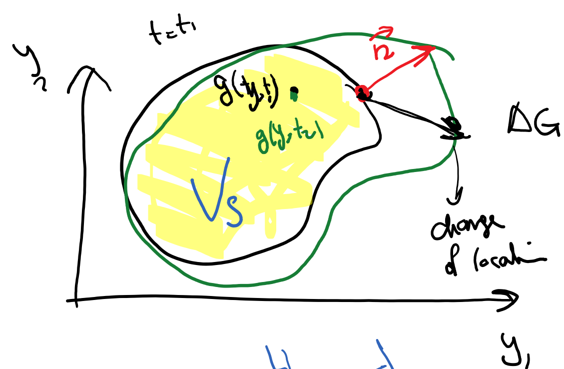
P_t domain of integration @ time t

An informal but insightful proof for line 3



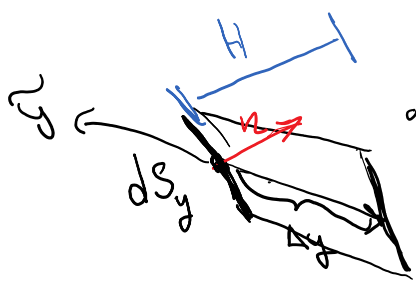
$G(t_1) = \int_{P_{t_1}} g(y,t_1)\,dV_y$

$G(t_2) = \int_{P_{t_2}} g(y,t_2)\,dV_y$

$\Delta G = G(t_2) - G(t_1) = \int_{V_s} [g(y,t_2) - g(y,t_1)]\,dV_y \Leftrightarrow$

$\Delta t \int_{P_t} \frac{\partial g}{\partial t}(y,t_1)\,dV_y$ + surface change cont.

cont 1



added volume

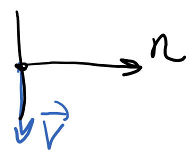
$\Delta V_y = dS_y \cdot H = dS_y\,(\vec{\Delta y}\cdot\vec{n})$

added integral $\approx \Delta V_y \cdot g(\tilde{y}, t_1)$

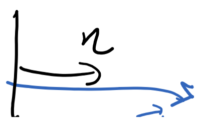
rate of change of integral from surface change $= \dfrac{\text{change of vol}}{\Delta t} = dS_y \left(\underbrace{\frac{\vec{\Delta y}}{\Delta t}}_{\vec{V}} \cdot n\right) g(y,t)$

Contribution from surface $dS_y \to$

$dS_y\, \vec{V}(y,t)\cdot\vec{n}\; g(y,t)$



$V\cdot n = 0$ change of vol zero



$V\cdot n = |V|$ highest rate of

$\vec{V} \cdot n = |V|$ highest rate of adding to volume

for the whole surface, contributions from change of volume are

$$\int_{\partial B_t} g(y,t)\, \vec{V} \cdot \vec{n}_y \, dS_y \qquad (C2)$$

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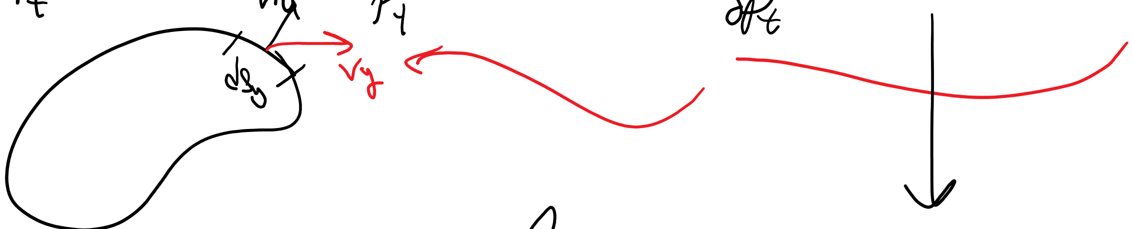
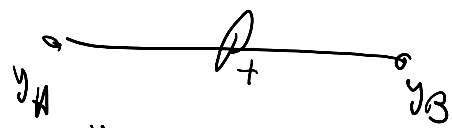
adding contributions 1 & 2

$$\frac{D}{Dt} \int_{P_t} g(y,t)\, dV_y = \int_{P_t} \frac{\partial g(y,t)}{\partial t}\, dV_y + \int_{\partial P_t} g(y,t)\, V(y,t) \cdot n_y \, dS_y$$



1D

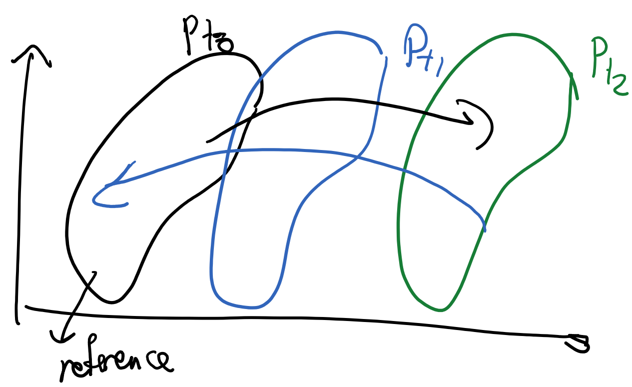


$$\frac{D}{Dt} \int_{y_A}^{y_B} g(y,t)\, dy = \int_{y_A}^{y_B} \frac{\partial g}{\partial t}\, dy + g(y_A,t)\, V_A \cdot \underset{-1}{n_A} + g(y_B,t)\, V_B \cdot \underset{+1}{n_B}$$

Formal Proof:

Eulerian expression

$$G(t) = \int_{P_t} g(y,t)\, dV_y$$



$x \longleftrightarrow y$ 1-1 map

$y = f(x,t)$

we express everything in terms of x

$$G(t) = \int \underbrace{g(y(x,t),t)}_{g(x,t)} \, dV \qquad ? \; dV_x = dV_y$$

we express everything in terms of x ~~y~~

$$G(t) = \int_{P_{t_0}} \overbrace{g(y(x,t),t)}^{g(x,t)} \underbrace{(J)\, dV_x}$$

$? \; dV_x = dV_y$

$? \quad = \frac{dV_y}{dV_x} = \det \underbrace{\nabla_y}_{F} {}_X$

$= \det F = J.$

$$\boxed{G(t) = \int_{P_{t_0}} g(x,t)\, J(x,t)\, dV_x}$$

Lagrangian expression

$$\boxed{\frac{DG(t)}{Dt} = \int_{P_{t_0}} \frac{D\, g(x,t)\, J(x,t)}{Dt}\, dV_x} \qquad \text{II}$$

these are all material time derivatives as quantities are expressed in x

↳ domain of integration does not change

$$\frac{DgJ}{Dt} = \frac{Dg}{Dt} J + g \frac{DJ}{Dt}$$

product der. rule

$$\frac{DJ}{Dt} = (\underset{\text{↳ Eulerian}}{\operatorname{div} \hat{v}})\, J$$

$$\frac{DgJ}{Dt} = \frac{Dg}{Dt} J + g \operatorname{div} \hat{v}\, J = \underbrace{\left(\frac{Dg}{Dt} + g \operatorname{div} \hat{v}\right) J}$$

$$\text{II} \quad \frac{DG}{Dt} = \int_{P_{t_0}} \frac{DgJ}{Dt}\, dV_x = \int_{P_{t_0}} \left(\frac{Dg}{Dt} + g \operatorname{div} \hat{v}\right) \underbrace{J\, dV_x}_{dV_y}$$

we can go back to y

$$\boxed{\frac{DG}{Dt} = \int_{P_t} \left(\frac{Dg}{Dt} + g \operatorname{div} \hat{v}\right) dV_y}$$

$$\underbrace{\frac{Dg}{Dt}}_{\text{material}} = \underbrace{\frac{\partial g}{\partial t} + \nabla g\, v}_{\text{spatial}} \qquad \text{III}$$

$$\frac{DG}{Dt} = \int \left[\frac{\partial g}{\partial t} + \nabla g\, v + g \operatorname{div} v\right] dV_y$$

$$\boxed{\frac{DG}{Dt} = \int \left[\frac{\partial g}{\partial t} + \nabla g \, v + g \, \mathrm{div} \, v \right] dV_y}$$

Line 1

I did all calculation for scalar g (g a vector is your HW)

for scalar g:

$$\nabla g \, v + g \, \mathrm{div} \, v = \frac{\partial g}{\partial y_i} v_i + g \, \underbrace{v_{i,i}}_{\frac{\partial v_i}{\partial y_i}} = \underbrace{(g v_i)_{,i}}_{\frac{\partial g v_i}{\partial y_i}} = \mathrm{div}(g v)$$

div → spatial divergence

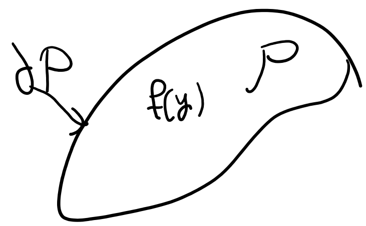
Replacing into eqn (3) we get

$$\boxed{\frac{DG}{Dt} = \int_{P_t} \left[\frac{\partial g}{\partial t} + \underbrace{\mathrm{div}(g v)}_{\text{full divergence}} \right] dV_y} \quad \text{(IV)}$$

Line 2 of Theorem 145

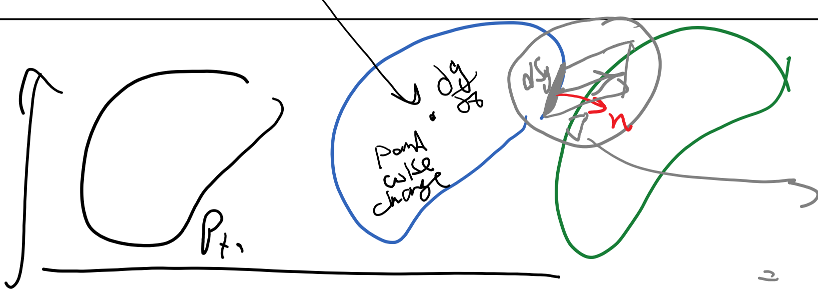
$$\int_P \mathrm{div} \, f \, dV_y = \int_{\partial P} f \cdot \underbrace{n \, dS_y}_{d\vec{S}_y}$$

Gauss Divergence theorem



if we apply Gauss theorem to eqn (IV) we get

$$\boxed{DG/Dt = \underbrace{\int_{P_t} \frac{\partial g}{\partial t} \, dV_y}_{} + \int_{\partial P_t} g \, \vec{v} \cdot \vec{n} \, dS_y} \quad \text{(V)}$$

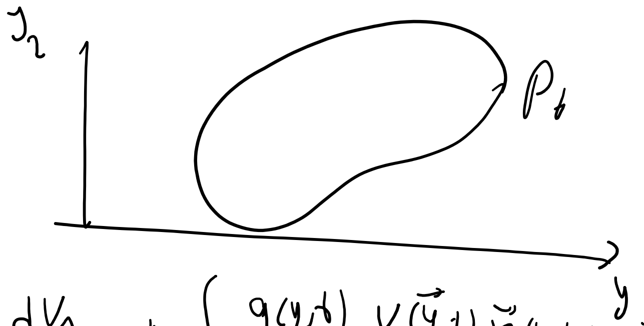


rate of change of vol = $V \cdot \vec{n} \, dS_y$

rate of added g $g(v \cdot n) \, dS_y$

Summary

$$G_t = \int_{P_t} g(y,t)\, dV_y$$



$$\frac{DG}{Dt} = \int_{P_t} \frac{\partial g(y,t)}{\partial t}\, dV_y + \int_{P_t} g(y,t)\, v(\vec{y},t)\, \vec{n}(y,t)\, dS_y$$

material rate

Eulerian

apply Gauss theorem

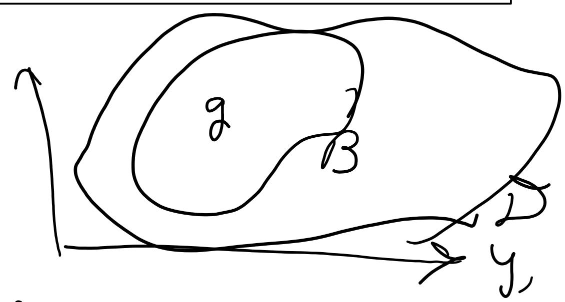
$$= \int \left[\frac{\partial g}{\partial t} + \mathrm{div}\,(g v)\right] dV_y$$

Recall Balance laws

$$\frac{D}{Dt} G = \int_B r_g\, dV_y - \int_{\partial B} f_g^y\, n\, dS_y$$

source term

outward spatial flux



$g = \rho$ Balance of Mass

$M = \int \rho\, dV$ $\qquad \frac{D}{Dt} M = 0 \qquad r_\rho = 0 \qquad f_\rho^y = 0$

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linear Momentum

$g = p = \rho v$

linear momentum density

$$\frac{DP}{Dt} = \sum F = \int_B \rho b\, dV_y + \int T \cdot n\, dS_y$$

cauchy stress

Energy only heat conduction effect

$$\mathcal{E} = \int_B e\, dV$$

$$\frac{D\mathcal{E}}{Dt} = \int_B Q\, dV_y - \int_{\partial B} q \cdot n\, dS_y$$

heat source

heat flux vector

B ↳ heat source $\qquad \partial \breve{B}$ ↳ heat flux vector

summary

g	r_g	f_g^y	Balance law name
ρ	0	0	Mass
$p = \rho \vec{v}$	ρb	$-T$	Linear momentum
$e_v = c_v T$	Q	q	Energy (only heat conduction)

Putting every thing together

(A) Balance law for body B_t $\qquad G = \int_{B_t} g \, dV_y$

$$\frac{DG}{Dt} = \int_{B_t} r^s \, dV_y - \int_{\partial B_t} f_g^y \cdot n_y \, dS_y$$

(B) Transport theorem

$$\frac{DG}{Dt} = \int_{B_t} \frac{\partial g}{\partial t} \, dV_y + \oint g v \cdot n_y \, dS_y$$

L3 of theorem 145

$$\int_{B_t} r_g \, dV_y - \int_{\partial B} f_g^y \, n \, dS = \int_B \frac{\partial g}{\partial t} \, dV + \int_{\partial B} g v \cdot n \, dS_y$$

$$\rightarrow \int_B \left(\frac{\partial g}{\partial t} - r_s \right) dV_y + \underbrace{\int_{\partial B} \left(g v + f_g^y \right) \cdot n \, dS}_{\text{divergence theorem}} = 0$$

divergence theorem

$$\int_B \left[\frac{\partial g}{\partial t} - r_s + \nabla_y \cdot (gv + f_g^y) \right] dV_y = 0$$

B arbitrary $\longrightarrow$ localization

$$\frac{\partial g}{\partial t} + \nabla_y \cdot F_g^y = r_s \qquad \text{PDE}$$

$$\underbrace{F_g^y}_{\text{total spatial flux}} = \underbrace{f_g^y}_{\text{"diffuse flux"}} + \underbrace{gv}_{\text{"advective flux"}}$$