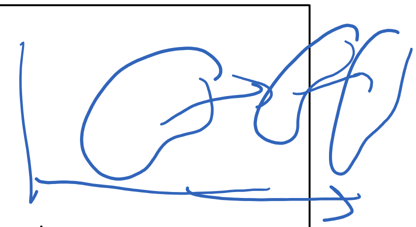
M2021/11/09

Tuesday, November 9, 2021 4:23 PM

(A) Balance law for $G = \int_{B_t} g \, dV_y$

body B_t



$$\frac{DG}{Dt} = \int_{B_t} r^s \, dV_y - \int_{\partial B_t} f_g^y \cdot n_y \, dS_y$$

(B) Transport theorem

$$\frac{DG}{Dt} = \int_{B_t} \frac{\partial g}{\partial t} \, dV_y + \int g v \cdot n_y \, dS_y$$

L3 of theorem 1.45

$$\int_{B_t} r^s \, dV_y - \int_{\partial B_t} f_g^y \cdot n_y \, dS_y = \int_{B_t} \frac{\partial g}{\partial t} \, dV_y + \int_{\partial B_t} g \, v \cdot n_y \, dS_y$$

$$\int_{B_t} \left(\frac{\partial g}{\partial t} - r^s \right) dV_y + \int_{\partial B_t} \left(f_g^y + g v \right) \cdot n_y \, dS_y = 0$$

→ If (can we always do it?) we can apply divergence theorem:

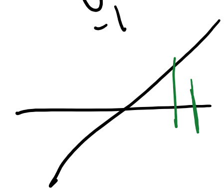
$$\int_{B_t} \Big\{ \underbrace{\frac{\partial g}{\partial t} - r^g + \nabla \cdot \big(\underbrace{f_g^y + g v}_{F_g^y} \big)}_{\text{integrand}} \Big\} dV_y = 0$$

(I)

this holds for any B_t

$$\int_{-1}^{1} x \, dx = 0$$

$\int (\ ) = 0$ → function is zero

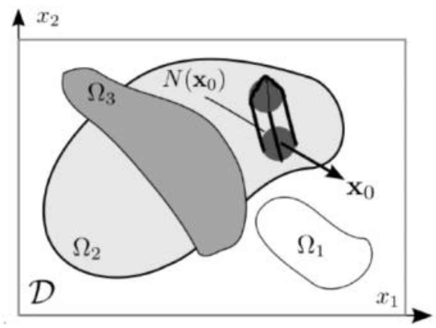


but $x \neq 0$

Localization theorem

Localization theorem states that if the integral of a continuus function is zero for all subsets of $\mathcal{D}$, then the function is zero:

$$\boxed{\forall \Omega \subset \mathcal{D}: \int_{\Omega} g(\mathbf{x})\, \mathrm{dv} = 0 \quad \Rightarrow \quad \forall x \in \mathcal{D}:\ g(\mathbf{x}) = 0} \qquad (21)$$



Let's assume $g(x_0) \neq 0$ (e.g., $g(x_0) > 0$). Since $g(\mathbf{x})$ is continuous, there is a neighborhood of $\mathbf{x}_0$ ($N(\mathbf{x}_0)$) that $g(x) > 0$. We choose an Ω that is only nonzero inside $N(\mathbf{x}_0)$. Then, $\int_{\Omega} g(\mathbf{x})\, \mathrm{d}V > 0$. Thus, $g(\mathbf{x}_0)$ cannot be nonzero and the function g is identically zero.

From I and localization theorem we obtain:

$$\frac{\partial s}{\partial t} + \nabla_y \cdot F_s^y = r_s^y$$

temporal flux density

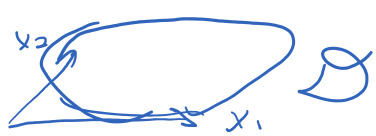
OUTWARD spatial flux density

source term

$$F_s^y = f^y + \underbrace{s \otimes v}_{\text{advective flux}}$$

non-convective or "diffusive" flux

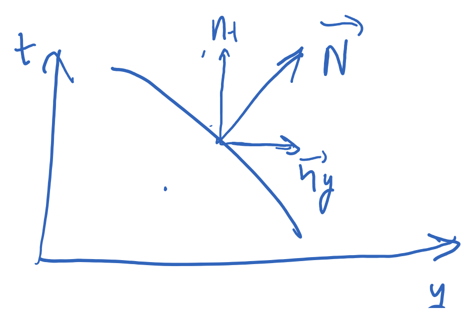
$\forall x \in \mathcal{D}$



Strong form (or PDE)

Some comments about balance laws:

A) Balance laws are general, whereas the PDE that we derive is not. The PDE only holds in the regions where the solution is smooth enough so we can apply the divergence theorem (when F_s^y is C^1 meaning that itself and its partial derivatives are continuous).



$[F_s^y]\, n_y + [s]\, n_t = 0$

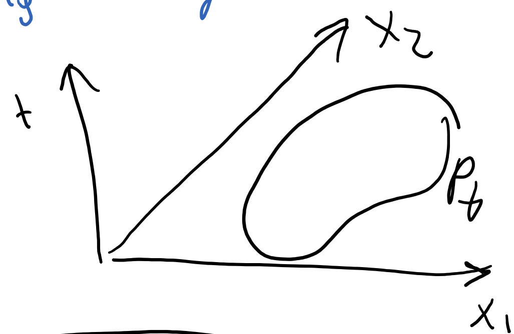
Also called Rankine Hugoniot condition in fluid mechanics

These jump conditions hold when the solution is not smooth enough

B. Static or steady state balance laws

balance law can be written as

$$\boxed{\begin{gathered} \underbrace{\frac{DS}{Dt}}_{0} = \int_{P_t} r_s \, dV_y - \int_{\partial P_t} f_y^y \, n \, dS_y \\ \int_{\partial P_t} f_s^y \, n \, dS_y = \int_{P_t} r_s \, dV_y \\ \underbrace{\int_{P_t} \nabla \cdot f_s^y \, dV_y}_{} = \int_{P_t} r_s \, dV_y \rightarrow \underbrace{\nabla_y \cdot f_y^s}_{} - r_s = 0 \end{gathered}}$$

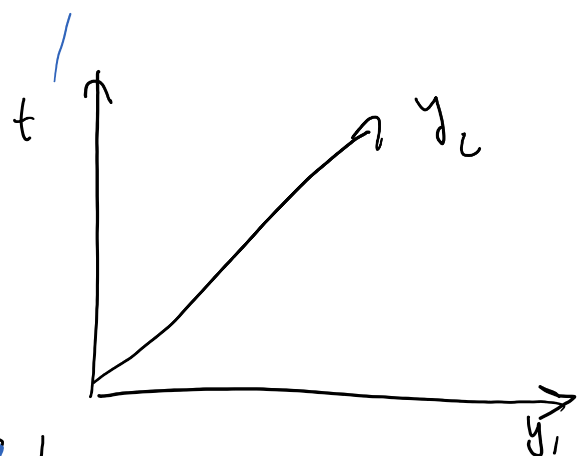


compare this with "dynamic balance law above

$$\frac{\partial s}{\partial t} + \underbrace{\nabla \cdot F_y^s - r_s}_{} = 0$$

added term for dynamic

$$F_y^s = f_y + s\, v$$



C. I want to write balance laws for general shapes in spacetime

spacetime coordinate = $[\vec{y}, t]$ so in 2D

$$\vec{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \hline t \end{bmatrix} \qquad F^S = \left[\begin{matrix} F_y^s \end{matrix} \right] \rightarrow \text{spatial part of it}$$

$$Y = \begin{bmatrix} \vdots \\ y^2 \\ \hline t \end{bmatrix}$$

$$F_{st}^{s} = \begin{bmatrix} F_y^{g} \\ \hline g \end{bmatrix} \begin{matrix} \rightarrow \text{spatial part of it} \\ \rightarrow \text{temporal part of it} \end{matrix}$$

F_{st}^{s} → space time flux density for s

$$\nabla_{st} = \begin{bmatrix} \nabla_y \\ \hline \frac{\partial}{\partial t} \end{bmatrix} \begin{matrix} \rightarrow \text{spatial grad} \\ \rightarrow \text{temporal grad(derivative with respect to time)} \end{matrix}$$

$$\nabla_{st} \cdot F_{st} = \begin{bmatrix} \nabla_y \\ \partial/\partial t \end{bmatrix} \cdot \begin{bmatrix} F_y^{g} \\ g \end{bmatrix} = \nabla_y \cdot F_y^{g} + \frac{\partial g}{\partial t} = r_g$$

a nice way to write spacetime balance law is

$$\boxed{\nabla_{st} \cdot F_{st}^{g} = r_g \quad \text{where} \quad \nabla_{st} = \begin{bmatrix} \nabla_y \\ \partial/\partial t \end{bmatrix} \quad F_{st}^{g} = \begin{bmatrix} F_y^{g} \\ g \end{bmatrix}}$$

PDE in ST

this looks like steady state strong form that we had a full divergence, but this is for dynamics & its a nicer representation of dynamic PDE above $\left(\frac{\partial g}{\partial t} + \nabla_y \cdot F_y^{g} - r^{g} = 0 \right)$

$$\int_{\Omega_t} \nabla_{st} \cdot F_{st}^{g} \, dV_{st} = \int_{\Omega_t} r_g \, dV_{st}$$

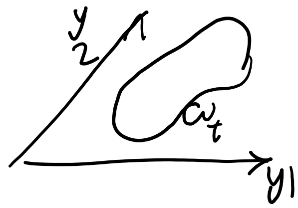
dynamics

$$\int F_{st}^{g} \cdot N_{st} \, dS_{st} = \int r_g \, dV_{st}$$

dynamics

$$\int_{\partial \Omega_{st}} f_{st} \cdot N_{st} \, dS_{st} = \int_{\Omega_t} r_g \, dV_{st}$$

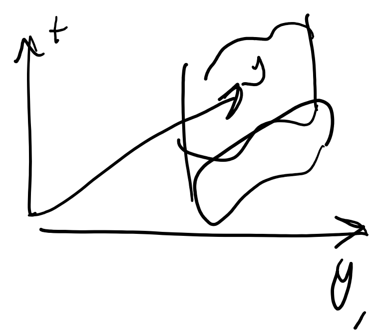
Static / Steady state

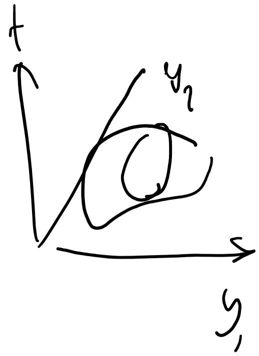


$$\int_{\partial \omega_t} f_y^g \cdot n_y \, dS_y = \int_{\omega_t} r_g \, dV_s$$

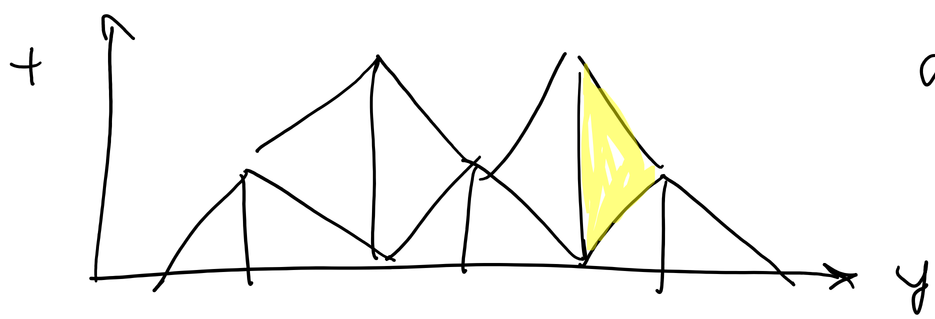
why space-time expression is good?

① It's more general





Application Spacetime FEMs / DG Methods



a SDG

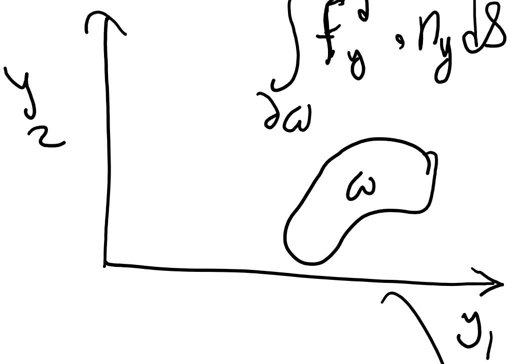
② Derivation of Jump (Rankine - Hugoniot conditions)

Balance law

Static

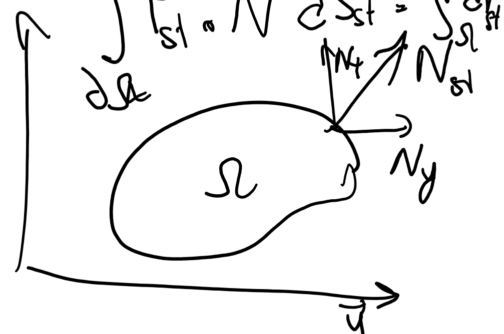
Dynamic

$$\int F_{st}^g \cdot N \, dS_{st} = \int r_g \, dV_{st}$$



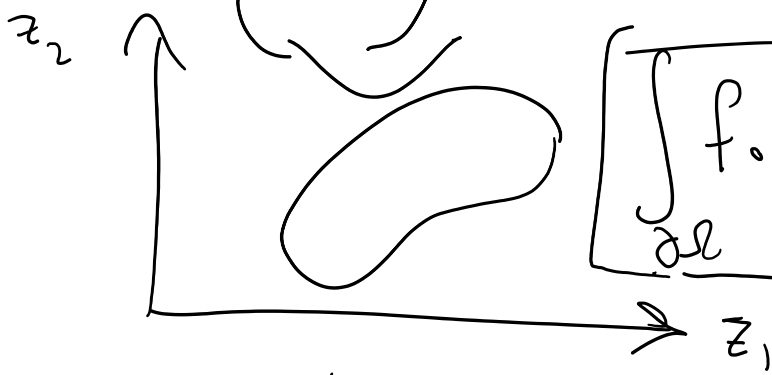
$$\int_{\partial\omega} f^g_y \cdot n_y \, dS = \int_\omega r^g \, dV$$

$$+ \int_{\partial\Omega} F^g_{st} \cdot N \, dS_{st} = \int_\Omega r^g_{st} \, dV_{st}$$



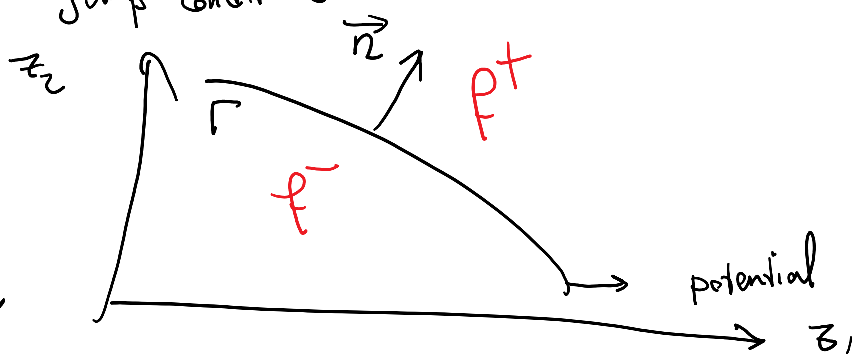
for example (y_1, y_2)

for both



$$\boxed{\int_{\partial\Omega} f \cdot n \, dS = \int_\Omega r \, dV} \quad (*)$$

derivation of jump conditions



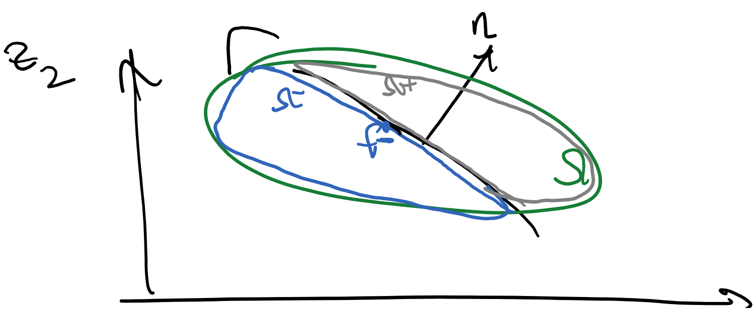
potential jump manifold

$[f] = f^+ - f^-$

what can we say about $[f]$

Jump condition $\boxed{[f] \cdot n = 0}$

How to prove this

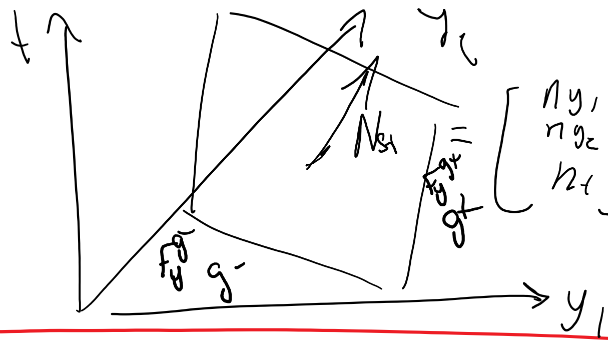


write (*) for Ω

− " (*) for Ω^-

− " " for Ω^+

How does the jump term look like in dynamics



$[F_{St}]\cdot N_{St} = 0$

$= \begin{bmatrix} n_{y_1} \\ n_{y_2} \\ n_t \end{bmatrix}$ $\vec{n}_y$

$\begin{bmatrix} [F_y^g] \\ [g] \end{bmatrix} \cdot \begin{bmatrix} \vec{n}_y \\ n_t \end{bmatrix} = 0$

R-H condition or jump condition for fluid $\boxed{[F_y^g]\cdot \vec{n}_y + [g]\, n_t = 0}$

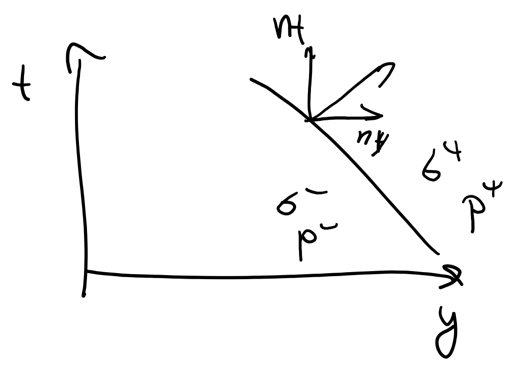
HW

$g = \rho v = p$

$F_y^g = -\sigma$ stress

solid mechanics $\hat{v} = 0$

$[-\sigma]\cdot \vec{n}_y + [p]\, n_t = 0$



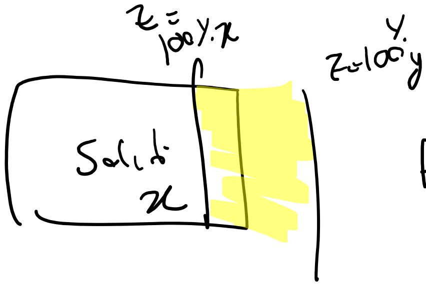
Balance laws for arbitrary coordinate system z

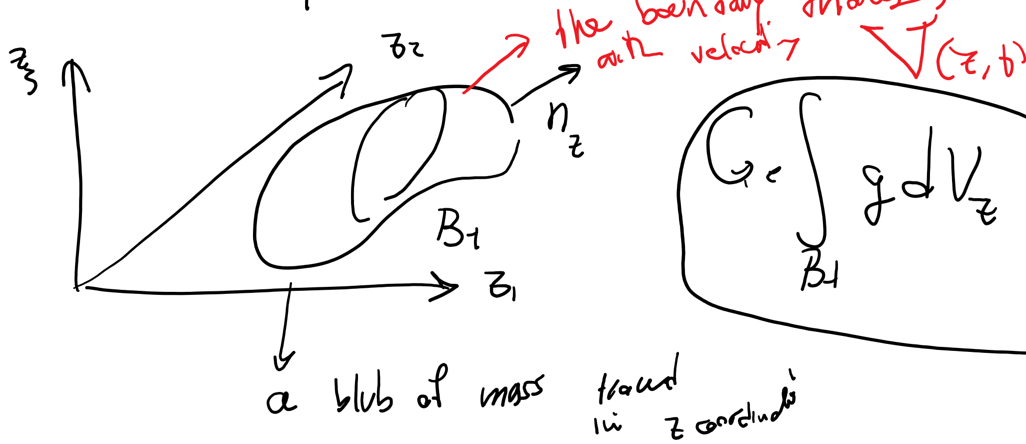
$z = x$ Lagrangian representation (good for solids)

$z = y$ Eulerian (" " fluids)

z in between

→ Arbitrary Lagrangian Eulerian (ALE) formulation





$$G = \int_{B_t} g \, dV_z$$

$$\left.\frac{DG}{Dt}\right|_{x\text{-fixed}} = \int_{P_t} r_g^z \, dV_z - \int_{\partial P_t} f_g(z,t)\, n_z \, dS_y$$

material rate

balance law

$$\left.\frac{DG}{Dt}\right|_{x\text{-fixed}} = \underbrace{\int_{P_t} \left.\frac{\partial g(z,t)}{\partial t}\right|_{z\text{-fixed}} dV_z}_{} + \int_{\partial P_t} g\, V(z,t)\, n_z \, dS_y$$

$$\int_{P_t} \left(\left.\frac{\partial g}{\partial t}\right|_{z\text{-fixed}} - r_g^z \right) dV_z + \int_{\partial P_t} \left(g \otimes V + f_g^z \right) n_z \, dS_y = 0$$

$$\int_{P_t} \left[\left.\frac{\partial g}{\partial t}\right|_{z\text{-fixed}} - r_g^z + \nabla_z \cdot \left(g \otimes V + f_g^z \right) \right] dV_z = 0$$

& P_t is arbitrary

$$\frac{\partial g(z,t)}{\partial t} - r_g^z(z,t) + \nabla_z \cdot \left(\underline{g \otimes V} + f_g^z\right)(z,t) = 0$$

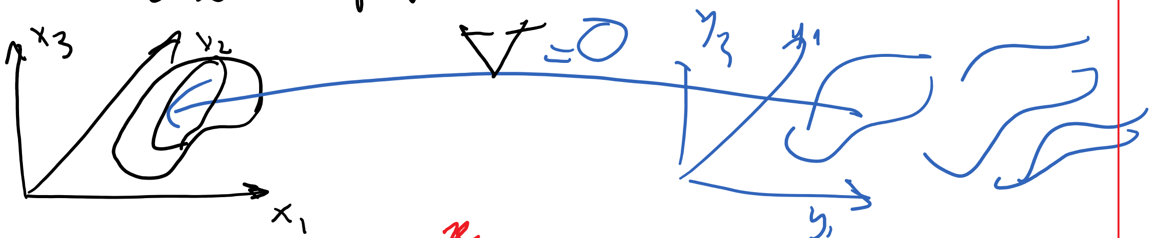
PDE for general z

Case 1 $z = y$ Eulerian $V = \hat{V}(y,t)$ actual velocity

$$\frac{\partial g(y,t)}{\partial t} - r_g^y(y,t) + \underbrace{\nabla_y \cdot}_{\text{div}} \left(g \otimes \hat{V}(y,t) + f_g^y\right)(y,t) = 0$$

we already derived this earlier today

case 2 $z = x$ Lagrangian



$$\frac{Dg(x,t)}{Dt} - r_g^x(x,t) + \nabla_x \cdot \left(f_g^x \right) = 0$$

→ Material derivative

$g \otimes V_x = 0$