CM2021/11/11

Thursday, November 11, 2021 4:30 PM

Conservation of mass:

$M_{t_1} = M_{t_2} = \dots$

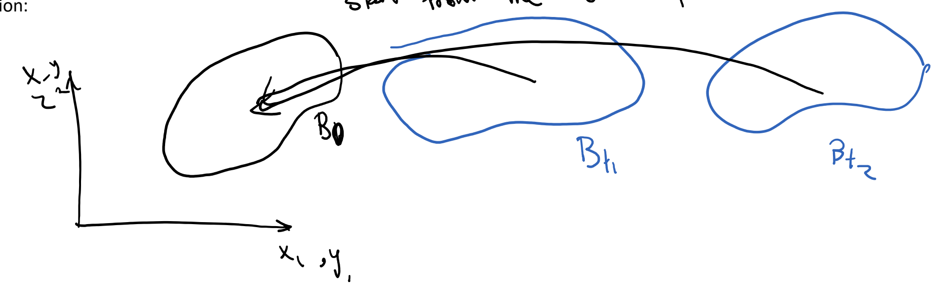
$$\frac{DM_P}{Dt} = 0 = \frac{D}{Dt}\int \rho \, dV_y = \int_{P_t} \overset{0}{\cancel{\rho_M}} \, dV_y + \int \overset{0}{\cancel{\rho_M^y}} \, dS_y = 0$$

ρ → mass density; ρ_M → mass source; ρ_M^y → mass spatial flux

$$\boxed{\frac{DM}{Dt} = 0 = \frac{D}{Dt}\int \rho \, dV_y} \quad 1$$

Lagrangian Interpolation:

start from the same position



$$\frac{DM}{Dt} = 0 \longrightarrow M(t) = M(0)$$

$$\int_{B_t} \hat{\rho}(y,t) \, dV_y = \int_{B_0} \rho_0(x) \, dV_x$$

change of variables
$dV_y = J \, dV_x$

$$\int_{B_0} \hat{\rho}(y,t) J \, dV_x = \int_{B_0} \rho_0(x) \, dV_x$$

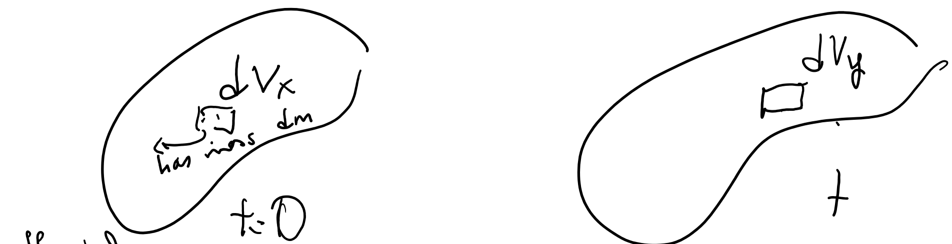
$$\longrightarrow \int_{B_0} \underbrace{[\hat{\rho}(y,t) J - \rho_0(x)]} \, dV_x = 0$$

B_0 → arbitrary → localization theorem →

$$\hat{\rho}(y,t) J - \rho_0(x) = 0$$

Lagrangian Mass conservation ② $\boxed{\rho J = \rho_0 \implies \rho(x,t) = \frac{\rho_0(x)}{J(x,t)}}$

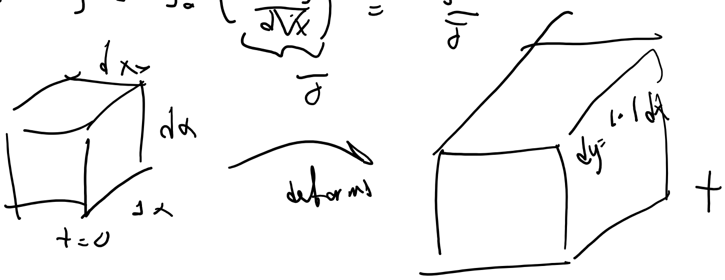
Intuitive way of looking at this



differential mass

$$dm = \rho_0\, dV_x = \rho\, dV_y$$

$$\rightarrow \rho = \rho_0 \underbrace{\left(\frac{dV_y}{dV_x}\right)}_{J}^{-1} = \frac{\rho_0}{J}$$



$$\rho(x,t) = \frac{\rho_0(x)}{1.1^3} = \frac{\rho_0}{J}$$

Interesting outcome of Lagrangian balance of mass:
Recall

Theorem 145 (Transport Theorem) *Let $g \in C^1(\Im, \Re)$ be a spatial scalar field. Then*

$$\begin{aligned}
\frac{D}{Dt}\int_{\mathcal{P}_t} g(\mathbf{y},t)\, dV_y &= \int_{\mathcal{P}_t}\left[\frac{\partial g}{\partial t}(\mathbf{y},t) + g_{,i}(\mathbf{y},t)\hat{v}_i(\mathbf{y},t) + g(\mathbf{y},t)\hat{v}_{i,i}(\mathbf{y},t)\right] dV_y \\
&= \int_{\mathcal{P}_t}\left\{\frac{\partial g}{\partial t}(\mathbf{y},t) + [g\hat{v}_i]_{,i}(\mathbf{y},t)\right\} dV_y \\
&= \int_{\mathcal{P}_t}\frac{\partial g}{\partial t}(\mathbf{y},t)\, dV_y + \int_{\partial\mathcal{P}_t} g(\mathbf{y},t)\,[\hat{\mathbf{v}}(\mathbf{y},t)\cdot\mathbf{n}(\mathbf{y},t)]\, dA_y,
\end{aligned}$$

$\nabla \cdot g\mathbf{v}$

I liked this :)

reduced transport

ρg

Theorem 151 (Reduced Transport Theorem) *Let $g \in C^1(\Im, \Re)$. Then*

$$\begin{aligned}
\frac{d}{dt}\int_{\mathcal{P}_t} g(\mathbf{y},t)\rho(\mathbf{y},t)\, dV_y &= \int_{\mathcal{P}_t}\left[\frac{\partial g}{\partial t}(\mathbf{y},t) + g_{,i}(\mathbf{y},t)\hat{v}_i(\mathbf{y},t)\right]\rho(\mathbf{y},t)\, dV_y. \\
&= \int_{\mathcal{P}_t}\left[\frac{\partial g}{\partial t}(\mathbf{y},t) + \nabla g(\mathbf{y},t)\cdot\hat{\mathbf{v}}(\mathbf{y},t)\right]\rho(\mathbf{y},t)\, dV_y
\end{aligned}$$

not very nice

Proof:
Smart proof

$$dm_x = \rho_0\, dV_x$$

$$\frac{D}{Dt}\int_{P_t} g\,\overline{\rho\, dV_y} = \frac{D}{Dt}\int_{P_t} g\, dm_y = \frac{D}{Dt}\int_{\underbrace{P_0}_{\text{constant @ time 0}}} g\, dm$$

$$\int \left(\frac{D}{Dt} g\right) dm \underset{=}{\text{go back to } P_0}$$

$$\int_{P_0} \left(\frac{D}{Dt} g\right) dm \overset{\text{go back to } P_t}{=}$$

constant

$$= \int \left(\frac{D}{Dt} g\right) \rho \, dV_y$$

$$\boxed{\begin{aligned} &\frac{D}{Dt} \int_{P_t} g \rho \, dV_y = \int_{P_t} \underbrace{\frac{D}{Dt}(g)}_{\frac{\partial g}{\partial t} + \nabla g \, \hat{v}} \rho \, dV_y \\ (3) \quad &= \int_{P_t} \left(\frac{\partial g}{\partial t} + \underbrace{\nabla g \ \hat{v}}_{\text{it's not a full } \nabla \cdot (\ ) \text{ not good}} \right) \rho \, dV_y \end{aligned}}$$

Balance of linear momentum

$$\frac{D}{Dt} \int \underbrace{\rho}_{\text{extra } \rho} \underbrace{\vec{v}}_{g} \, dV_y$$

Balance of mass in Eulerian framework

$$\frac{DM_{P_t}}{Dt} = \int_{P_t} \cancelto{0}{r_M} \, dV_y + \int_{\partial P_t} \cancelto{0}{f^y_M} \, dS_y = 0$$

$$\frac{DM}{Dt} = \frac{D}{Dt} \int_{P_t} \rho \, dV_y = \int_{P_t} \frac{\partial \rho}{\partial t} \, dV_y + \int_{\partial P_t} \rho \, \hat{v} \cdot n_y \, dS_y$$

$$= 0$$

$$\int_{P_t} \frac{\partial \rho}{\partial t} dV_y + \int_{\partial P_t} (\rho \hat{v}) \cdot n_y \, dS_y = 0$$

$$\int_{P_t} \frac{\partial \rho}{\partial t} dV_y + \int_{P_t} \nabla \cdot \rho \hat{v} \, dV_y = 0$$

P_t $\quad \downarrow \quad \mathcal{P}_t$

$$\int_{\partial P_t} \left[\frac{\partial \rho}{\partial t} + \underbrace{\nabla_y \cdot}_{\text{div}} (\rho \hat{V})\right] dV_y = 0 \longrightarrow \text{Localization}$$

Balance of mass in Eulerian framework

$$\frac{\partial \rho}{\partial t} + \underbrace{\text{div}}_{\nabla_y} (\underbrace{\rho \hat{V}}_{\text{spatial flux}}) = 0 \qquad (4)$$

$s = \rho$

$f^y = 0$

$r^y = 0$

$$\frac{\partial \rho}{\partial t} + \nabla_y \cdot \rho V = 0$$

Note: last time we had

$$\underbrace{\frac{\partial s}{\partial t}}_{\text{temporal flux density}} + \nabla_y \cdot \underbrace{F_s^y}_{\text{OUTWARD spatial flux density}} = \underbrace{r_s^y}_{\text{source term}}$$

$$F_s^y = \underbrace{f^y}_{\text{non-convective}} + \underbrace{s \otimes V}$$

Equation (4) can be written in a variety of forms

$$\frac{\partial \rho}{\partial t} + \nabla_y \cdot (\rho V) = 0$$

$$\nabla_y \cdot \rho V = \frac{\partial}{\partial y_i} (\rho \hat{V}_i) = \underbrace{\frac{\partial \rho}{\partial y_i} V_i}_{\nabla \rho\, V} + \underbrace{\rho \frac{\partial V_i}{\partial y_i}}_{\rho\, \text{div} V}$$

$$\frac{\partial \rho}{\partial t} + \underbrace{(\nabla \rho\, V + \rho\, \text{div} V)}_{\nabla_y \cdot \rho V} = 0$$

$$\left(\frac{\partial \rho}{\partial t} + \nabla \rho\, V\right) + \rho\, \text{div} V = 0$$

$$\frac{D\rho}{Dt} + \rho\, \text{div}\, V = 0$$

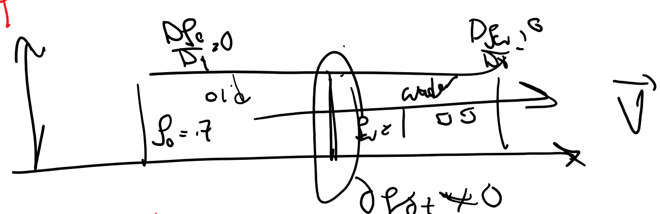
$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \rho V = \frac{D\rho}{Dt} + \rho\, \text{div}\, V = 0}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho v = \frac{D\rho}{Dt} + \rho \operatorname{div} v = 0$$

air . compressible ρ changes in time

water or many fluids are <u>nearly</u> incompressible . Means that their density can be considered constant in time.

$$\frac{D\rho}{Dt} = 0 \quad \text{for incompressible fluid}$$



$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho v = \cancel{\frac{D\rho}{Dt}} + \rho \operatorname{div} v = 0$$

incompressible $\cancel{\frac{D\rho}{Dt}} = 0 \longrightarrow \operatorname{div} \hat{v} = 0$

$$\frac{\partial \hat{v}_1(y,t)}{\partial y_1} + \frac{\partial \hat{v}_2(y,t)}{\partial y_2} + \frac{\partial \hat{v}_3(y,t)}{\partial y_3} = 0$$

Summary: Conservation of mass

Lagrangian $\boxed{\rho(x,t) = \frac{\rho_0(x)}{J(x,t)}}$

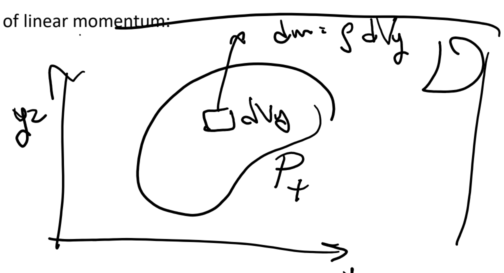
Eulerian $\frac{\partial \hat{\rho}(y,t)}{\partial t} + \operatorname{div}(\rho \hat{v}(y,t)) = \frac{D\hat{\rho}(y,t)}{Dt} + \rho \operatorname{div} \hat{v} = 0$

Incompressibility

$\rho = \rho_0 \Leftrightarrow J(x,t) = 1$

$\frac{D\rho}{Dt} = 0 \Rightarrow \operatorname{div} \hat{v} = \frac{\partial \hat{v}_i}{\partial y_i} = 0$

Balance of linear momentum:



$$\Sigma F = \frac{DP}{Dt}$$

linear momentum $\sim P = m\vec{V}$

$$P = \int_{P_t} (dm)\vec{v} = \int_{P_t} \rho \vec{v} \, dV_y$$

$\rho\vec{v} \to$ p (or m in fluid mechanics) linear momentum density

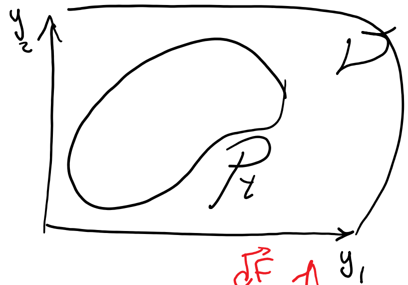
$P = \int p \, dV_y \qquad \frac{DP}{}$

$$\boxed{P = \int_{P_t} p \, dV_y \qquad \frac{DP}{Dt} = \sum \text{Forces acting in } P_t}$$

$\vec{p} = \rho \vec{v}$

$$\frac{DP}{Dt} = \sum \text{Forces} = F_{volumetric} + F_{surface}$$

$$= \int_{P_t} \vec{b} \underbrace{dm}_{\rho dV_y} + \int_{\partial P_t} \vec{t} \, dA_y$$

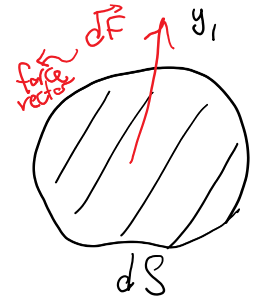


$\vec{b}$: body force = force per unit mass

e.g. for gravity $b = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}$

$\vec{b}$ has unit of acceleration

$\vec{t}$



$\vec{t} = \lim_{dS \to 0} \frac{d\vec{F}}{dS}$

force on ~~sur~~ surface is extensive

$\Rightarrow$ density of $\vec{F}$ = traction

$$\frac{DP}{Dt} = \frac{D}{Dt}\int_{P_t} p \, dV_y = \int_{P_t} \rho b \, dV_y + \int_{\partial P_t} \vec{t} \, dS_y$$

$$\vec{t} = T n_y$$

vector, vector

T: Cauchy stress tensor

2nd order tensor

for now take it as it is

$$\rightarrow \frac{DP}{Dt} = \int_{P_t} \underbrace{\rho b}_{r_P^y} \, dV_y - \int_{\partial P_t} \underbrace{(-T)}_{f_P^y} \cdot n_y \, dS_y$$

e.g. for g: $\frac{DG}{Dt} = \frac{D}{Dt}\int_{P_t} g \, dV_y = \int_{P_t} \underbrace{r_g^y}_{\text{source term for } g} dV_y - \int_{\partial P_t} \underbrace{f_g^y}_{\text{"diffuse"}} \cdot n_y \, dS_y$ spatial flux

recall

$$\frac{DG}{Dt} = \int_{P_t} \frac{\partial g}{\partial t} dV_y + \int_{\partial P_t} g v \cdot n_y \, dS_y$$

So for $g = \vec{p}$ we'll have

$$\int_{P_t} \rho b \, dV_y + \int_{\partial P_t} T \cdot n_y \, dS_y = \int_{P_t} \frac{\partial p}{\partial t} dV_y + \int p \, \vec{v} \cdot \vec{n}_y \, dS_y$$

$$p \, \vec{v} \cdot \vec{n} = (p_i e_i)(v_j n_j) = (p_i v_j n_j) e_i$$

$$\boxed{\text{equal}} \quad (\vec{p} \otimes \vec{v}) \vec{n} = ((p \otimes v)_{ij} e_i \otimes e_j)(n_k e_k) = (p_i v_j e_i \otimes e_j)(n_k e_k)$$

$$= p_i v_j n_k e_i \underbrace{(e_j \cdot e_k)}_{\delta_{jk}} = (p_i v_j n_j) e_i$$

$$\int_{P_t} \left(\frac{\partial p}{\partial t} - \rho b \right) dV_y + \int_{\partial P_t} (p \otimes v - T) \cdot n_y \, dS_y = 0$$

$$\int_{P_t} \left\{ \frac{\partial p}{\partial t} - \rho b + \operatorname{div}(p \otimes v - T) \right\} dV_y = 0$$

Localization

Balance of linear momentum / equation of motion (EOM)

$$\frac{\partial p}{\partial t} - \rho b + \operatorname{div}(\underbrace{\vec{p} \otimes \vec{v}}_{\text{density}} - \vec{T}) = 0$$

vector $\leftarrow$ $p = \rho \vec{v}$

$p = \rho \vec{v}$ linear momentum density — 2nd order tensor — $\vec{m}$ is also used

T = Cauchy stress (σ is often used)

⑤

Eulerian framework

This equation is the balance of linear momentum in **current configuration**:

- We need to use the current configuration to express balance of linear momentum (forces) because forces are applied in the current configuration.

conservation of mass

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\underbrace{\overset{p}{\widetilde{\rho v}}}_{\text{vector}}) = 0$$

scalar

$$\frac{\partial g}{\partial t} + \nabla_y \cdot \underbrace{F_g}_{\text{total}} = r_g^y$$

1 tensorial order higher than g