CM2021/11/16

Tuesday, November 16, 2021 4:31 PM

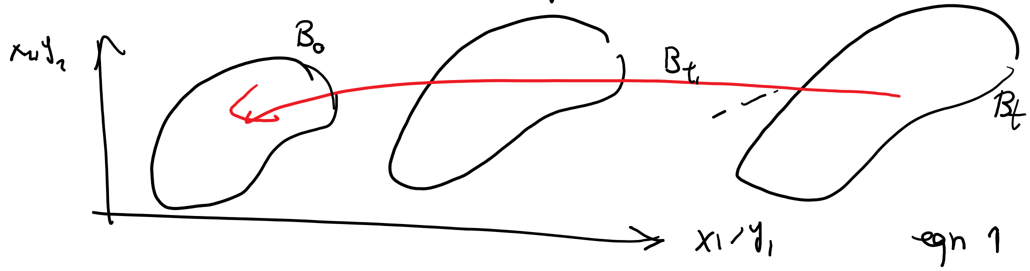
Balance of linear momentum / equation of motion (EOM)

$$\frac{\partial \vec{p}}{\partial t} - \rho \vec{b} + \operatorname{div}\left(\underbrace{\vec{p}\otimes\vec{v} - \bar{\bar{T}}}_{\text{flux density}}\right) = \vec{0} \qquad (1)$$

$$\frac{dP}{dt} = \sum F$$

- we sum the forces in the current configuration
- we follow a blob of mass ($\mathcal{B}_t$)

Goal: Express (1) in Lagrangian framework.



eqn 1 is written in y

Balance law was

$$\underbrace{\frac{DP}{Dt} = \frac{D}{Dt}\int_{\mathcal{B}_t} p\, dV_y}_{(1)} = \underbrace{\int_{\partial\mathcal{B}_t} T n_y\, dA_y}_{(2)} + \underbrace{\int_{\mathcal{B}_t} \rho b\, dV_y}_{(3)}$$

$$(1)\ \frac{D}{Dt}\int_{\mathcal{B}_t} \rho v\, dV_y = \int v \underbrace{(\rho\, dV_y)} = \frac{D}{Dt}\int_{\mathcal{B}_0} v \rho_0\, dV_x$$

side note: reduced transport

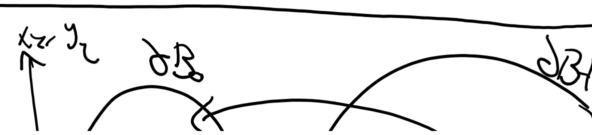
$$\frac{D}{Dt}\int g \underbrace{\rho\, dV_y}_{dm} = \int \frac{Dg}{Dt} \underbrace{(dm)}_{\rho_0 dV_x}$$

$$= \int_{\mathcal{B}_0} \frac{D}{Dt}\underbrace{\left[v(x,t)\, \rho_0(x)\right]} dV_x$$

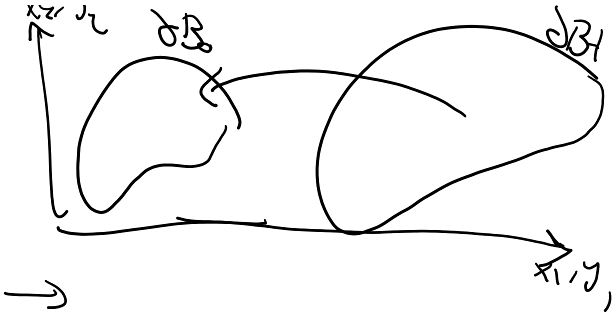
$$P_0(x,t) = \rho_0(x)\, v(x,t)$$

linear momentum density in reference configuration

(2) $\int T_0 n_x\, dA_y$



(2) $$\int_{\partial B_t} T \cdot \underbrace{n_y \, dA_y}_{\overrightarrow{dA_y}}$$



$$\overrightarrow{dA_y} = J F^{-t} \, \overrightarrow{dA_x}$$

we showed this relation in kinematics section

$$\int_{\partial B_t} T \, \overrightarrow{dA_y} = \int_{\partial B_0} T (\underbrace{J}_{\text{scalar}} F^{-t}) \, \overrightarrow{dA_x}$$

$$= \int_{\partial B_0} (J \, T F^{-t}) \cdot \underbrace{n_x \, dA_x}_{\overrightarrow{dA_x}}$$

$$\boxed{\begin{gathered} \underbrace{S(x,t)}_{\text{Piola Kirchhoff I stress tensor}} = J(x,t) \, \underbrace{T(y(x,t),t)}_{\text{Cauchy stress}} \, F^{-t}(x,t) \\ \text{PK-I} \\ \int_{\partial B_t} T \, \overrightarrow{dA_y} = \int S \, \overrightarrow{dA_x} \end{gathered}}$$

(3) $$\int_{B_t} \rho b \, dV_y = \int_{B_t} b \, (\overbrace{\rho \, dV_y}^{dm \,=\, \rho_0 \, dV_x}) = \int b \, \rho_0 \, dV_x$$

Summary:

Balance of linear momentum

$$\frac{D}{Dt} \int_{B_t} \rho \, dV_y = \int_{\partial B_t} T \, \overbrace{n_y \, dA_y}^{\overrightarrow{dA_y}} + \int_{B_t} \rho b \, dV_y$$

(1) (2) (3)

Spatial (Eulerian)

$$\frac{D}{Dt}\int_{B_t}\rho\,\vec{v}\,dv_y = \int_{\partial B_t}\sigma\,n_y\,da_y + \int_{B_t}\rho b\,dv_y$$

Spatial (Eulerian) rep.

$$\text{②}\quad \frac{D}{Dt}\int_{B_0} P_0\,dV_X = \int_{\partial B_0} S\,\underbrace{n_X\,dA_X}_{d\vec{A}_X} + \int_{B_0}\rho_0 b\,dV_X$$

Referential (Lagrangian) rep.

$$P_0 = \rho_0\vec{V}\,,\quad S = J\,T\,F^{-t}\quad \text{PK-I}$$

Obtaining the strong form of bal. of. lin mom. in reference configuration

$$\underbrace{\frac{D}{Dt}\int_{B_0} P_0\,dV_X}_{} = \underbrace{\int_{\partial B_0} S\,n_X\,dA_X}_{} + \int_{B_0}\rho_0 b\,dV_X$$

$$\int_{B_0}\left(\frac{D}{Dt}P_0\right)dV_X = \int_{B_0}\underbrace{\mathrm{Div}}_{\nabla_{X^0}}\,S\,dV_X + \int\rho_0 b\,dV_X$$

$$\forall B_0:\quad \int_{B_0}\left[\frac{D}{Dt}P_0 - \mathrm{Div}\,S - \rho_0 b\right]dV_X = 0$$

$\longrightarrow$ Localization $\quad \dfrac{DP_0}{Dt} - \mathrm{Div}\,S - \rho_0 b = 0$

$$\frac{DP_0}{Dt} = \frac{D\,\rho_0(X)\,V(X,t)}{Dt} = \rho_0(X)\,\frac{\overbrace{DV(X,t)}^{\text{acceleration}}}{Dt} = \rho_0\,\underbrace{a(X,t)}_{\ddot{U}(X,t)}$$

EOM in Lagrangian form is

$$\underbrace{\frac{D}{Dt}P_0}_{\rho_0 \frac{Dv}{Dt} = \rho a} - \mathrm{Div}\, S = \rho_0 b$$

$$\mathrm{Div}\, S = S_{ij,j}\, e_i \qquad \begin{bmatrix} S_{11,1} + S_{12,2} + S_{13,3} \\ S_{21,1} + S_{22,2} + S_{23,3} \\ S_{31,1} + S_{32,2} \quad S_{33,3} \end{bmatrix}$$

$$\rho_0 \ddot{u}_1 - (S_{11,1} + S_{12,2} + S_{13,3}) = \rho_0 b_1$$

$$\rho_0 \ddot{u}_2 - (S_{21,1} + S_{22,2} + S_{23,3}) = \rho_0 b_2$$

$$\rho_0 \ddot{u}_3 - (S_{31,1} + S_{32,2} + S_{33,3}) = \rho_0 b_3$$

Interpretation of S:

$$\frac{DP}{Dt} = F_{surface} + F_{body}, \quad F_{surface} = \int_{\partial B_t} T\, n_y\, dA_y$$

$$\vec{dA_y} = n_y\, dA_y$$

$$\vec{t}_y = T n_y \quad (\vec{dF_y} = T\, \vec{dA_y})$$

$$F_{surface} = \int S \underbrace{n_x dA_x}_{\vec{dA_x}}$$

$$\vec{dF}_y = \underbrace{S\, \vec{dA}_{\cancel{x}}}$$

Cauchy stress

— maps y-surface-differential to y-force //

= y normal vector to y-traction

PK-I

maps X-surface differential to y-force //

X-normal vector to y-force

$$\boxed{S = J\, T\, F^{-t}}$$

$$T$$

balance of angular momentum

S^t ... $J F^{-1} T^t$ ...

$$S^t = J F^{-1} T^t = J F^{-1} T^t \neq S$$

S is <u>not</u> symmetric :(

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$ 9-indep. values

balance of angular momentum

$\rightarrow T_{ij} = T_{ji}$ $(T = T^t)$

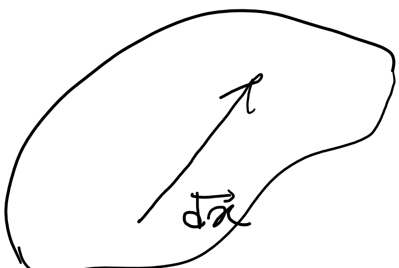
T is symmetric

$$T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ & T_{22} & T_{23} \\ \text{sym} & & T_{33} \end{bmatrix}$$

6-indep para.

Idea: How about we map surface differential in x to FORCE **DIFFERENTIAL in x**?

traction t or force differential dF is a vector



vector in x coordinate

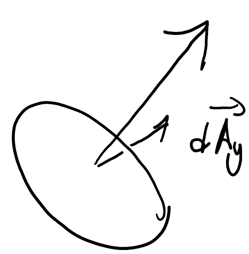
$\underset{\sim}{F}$

$\vec{dy} = \nabla_{y/x} \vec{dx}$

vector transformation between x & y

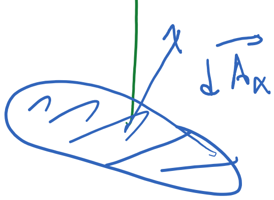
$$\boxed{\begin{aligned} \vec{dy} &= F \vec{dx} \\ \vec{dx} &= F^{-1} \vec{dy} \end{aligned}}$$

dF_y ✓ $d\vec{F_y}$



what is dF_x

$d\vec{F_x} = F^{-1} dF_y$



$$\left.\begin{aligned} d\vec{F_y} &= S\, dA_x \\ d\vec{F_x} &= F^{-1} d\vec{F_y} \end{aligned}\right\} \longrightarrow d\vec{F_x} = F^{-1} S\, d\vec{A_x}$$

$$\left.\begin{aligned} &\ldots \, d A_x \\ &d\vec{f}_x = F^{-1} d\vec{f}_y \end{aligned}\right\} \longrightarrow \quad \begin{aligned} &df_x = F^{-1}\, \sigma \, dA_x \\ &d\vec{f}_x = \underbrace{\left(J F^{-1} T F^{-t}\right)}_{P} d\vec{A}_x \end{aligned}$$

(PK-II)

PK-II maps $d A_x \longrightarrow d\vec{f}_x$

or $\cancel{\ }$ $\vec{n}_x \longrightarrow \vec{t}_x$

$$P = F^{-1} \sigma = J F^{-1} T F^{-t}$$

$$P^t = J (F^{-t})^t T^t (F^{-1})^t = J F^{-1} T F^{-t} = P$$

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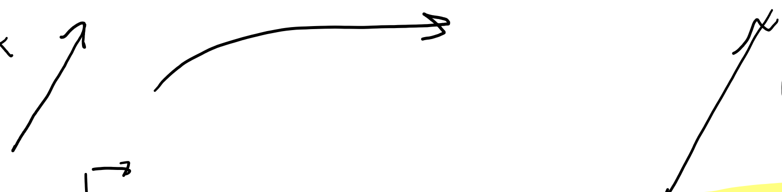
vectors:

forces, traction, line segment ($d\vec{y}$), ...

referential



t_x

$t_y = F t_x$

Example $d\vec{x}$

$d\vec{y} = F d\vec{x}$

COvectors (surfaces ...)

n_x

$d\vec{A}_x = n_x \, dA_x$

$d\vec{A}_y = n_y \, dA_y$

$d\vec{A}_y = J F^{-t} d\vec{A}_x$

Summary of stresses

stress tensor maps normal vector to traction

Cauchy T $\qquad \vec{n}_y \xrightarrow{T} \vec{t}_y$

PK-I S $\qquad n \xrightarrow{S}$

PK-I $\quad S$

PK-II $\quad P$

$n_x \xrightarrow{S}$

$n_x \xrightarrow{P} t_x$

Lagrangian EOM $\quad \rho_0 a - \mathrm{Div}\, S = \rho_0 b$

or $\quad \rho_0 \ddot{u} - \mathrm{Div}(FP) = \rho_0 b$

$S = J T F^{-t}$

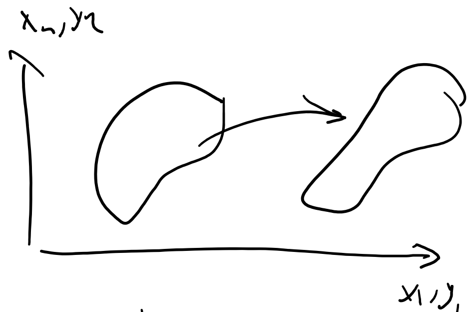
$P = J F^{-1} T F^{-t}$

P is better because it's sym.

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ \text{sym} & P_{22} & P_{23} \\ & & P_{33} \end{bmatrix}$$

Balance of energy (hand-out will be given for Thermodynamics laws I, II)
Thermodynamics law I

$\mathcal{E}$ total energy for body B_t



$$\frac{D\mathcal{E}}{Dt} = \frac{D}{Dt}\int_{B_t} e \, dV_y = \int_{B_t} r_e^y \, dV_y - \int_{\partial B_t} f_e^y \cdot dA_y$$

e: volumetric energy density (scalar)

$$e = \underbrace{\underbrace{\tfrac{1}{2}\rho\, \vec{v}\cdot\vec{v}}_{\text{kinetic energy density}} + \underbrace{U(F, \dot{F})}_{\text{strain-based energy density}}}_{\text{Mechanical}} + \underbrace{\underbrace{c_v}_{\text{volumetric capacity}} T}_{\text{thermal}}$$

internal energy from temperature

$+ \frac{1}{2}\left(E\cdot D + H\cdot B \right)$

thermal

$$+\frac{1}{2}(E\cdot D + H\cdot B)$$

electric field flux

magnetic field flux

$$r_\mathcal{E}^y = \rho\, b \cdot \vec{v} \qquad + Q$$

$$+ \underbrace{E \cdot J}_{\text{Joule's heating}}$$ → electric current density

$$\rho_\mathcal{E}^y = -\sigma \cdot v \qquad + q \qquad + \underbrace{E \times H}_{\text{Poynting vector}}$$

outward heat flux

Simplified case: Ignore Mechanical & EM contributions

$$\boxed{\begin{aligned} e_y &= C_v T \\ r_e &= Q \\ \rho_e^y &= q \end{aligned}} \longrightarrow \frac{\partial C_v T}{\partial t} + \operatorname{div} q = Q$$

no advection

if assume $q = -k\nabla T$ Fourier law

→ parabolic PDE