CM2021/11/18

Thursday, November 18, 2021 4:31 PM

Navier-Stokes equations:

Balance of energy:

volumetric energy density

energy source term

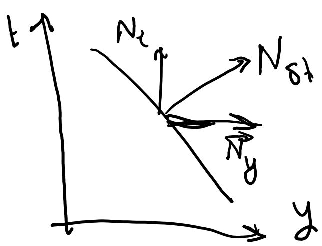
$$\underbrace{\frac{D}{Dt}\int_{B_t} e\, dV_y}_{\mathcal{E}} = \int_{B_t} \frac{\partial e}{\partial t}\, dV_y + \int_{\partial B_t} e\,\hat{v}\cdot n\, dA_y = \int_{B_t} r_e\, dV_y - \int_{\partial B_t} f_e^y \cdot n\, dA_y$$

outward flux of energy

→ as before using divergence theorem, we'll derive STRONG form of balance of energy

$$\frac{\partial e}{\partial t} + \mathrm{div}\left(F_e^y\right) = r_e^y \qquad \text{PDE}$$

$$F_e^y = \underbrace{f_e^y}_{\text{diffusive part}} + \underbrace{e\hat{v}}_{\text{advective part}}$$



$$F_{st} = \begin{bmatrix} F_e^y \\ e \end{bmatrix}$$

$$0 = [F_{st}]\cdot N_{st} = [F_e^y]\cdot \vec{N}_y + [e]\cdot N_t = 0 \quad \text{Jump condition}$$

HW 7 problem 1

we need to augment this with balance of mass & lin. momentum to have Navier Stokes equations

PDEs for N-S

1. Balance of Mass $\quad \frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\hat{v}) = \frac{D\rho}{Dt} + \rho\,\mathrm{div}\,\hat{v} = 0$

2. Balance of linear momentum (vector equation) $\quad \frac{\partial p}{\partial t} + \mathrm{div}\,(p \otimes v - T) = \rho b$

   $p = \rho\hat{v}$ (m is also used for p)

3. Balance of energy $\quad \frac{\partial e}{\partial t} + \mathrm{div}\,(ev - f_e^y) = r_e^y$

other equations — Balance of angular momentum $T = T^t$

— Constitutive equations ($T = ?$ how are
$e = ?$ these obtained)

+ Jump conditions

While solid mechanics equations can also be written in Eulerian framework, we often write them in Lagrangian framework:

1. Balance of mass -> nothing $\rho(x,t) = \frac{\rho_0(X)}{J(X,t)}$

    Do nothing:)

2. Balance of energy: this is automatically satisfied if we have only mechanical effects & linear momentum is satisfied (Cauchy stress is symmetric)

3. The only non-trivial equation is balance of linear momentum

$$\boxed{\frac{Dp_0}{Dt} + \mathrm{Div}\, S = \rho_0 b} \quad (i)$$

finite deformation "general"

$p_0 = \rho_0(X)\, v(X,t)$

P-J $S = J\, T\, F^{-t}$

infinitesimal theory $H = \nabla u/_x = O(\varepsilon)$

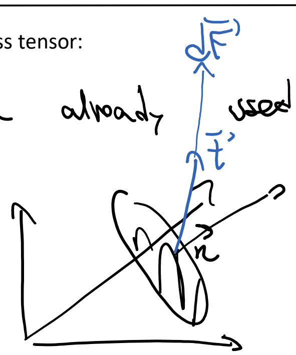
$$S = T + O(\varepsilon)$$
$$P = T + O(\varepsilon)$$

$$\boxed{\frac{Dp_0}{Dt} + \mathrm{Div}\, T = \rho_0 b} \quad (ii) \qquad S \approx P \approx T$$

infinitesimal limit

Traction and stress tensor:

I have already used this $\vec{t} = \vec{\sigma} \cdot \vec{n}$ but have not proved it!



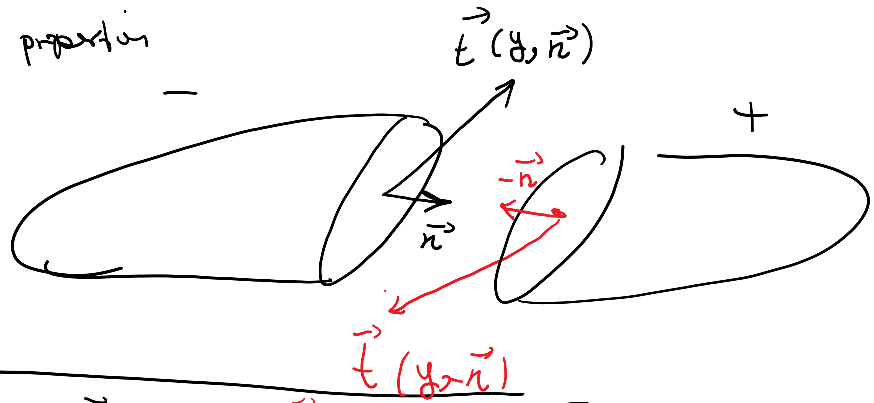
$\vec{dA} = n\, dA$

$$\vec{t} = \lim_{dA \to 0} \frac{d\vec{F}}{dA}$$

$\underbrace{\hat{t}(y,t)}_{\text{traction is expressed in current}}$

configuration

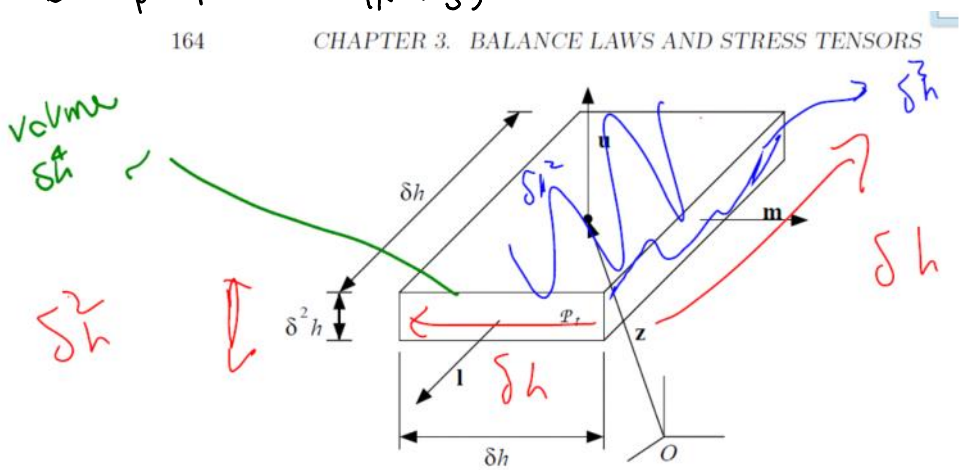
two properties



$$\boxed{\vec t(y,\vec n) = -\vec t(y,-\vec n)} \quad (\star)$$

Proof: Go back to problems 1 & 2 in HW 1:

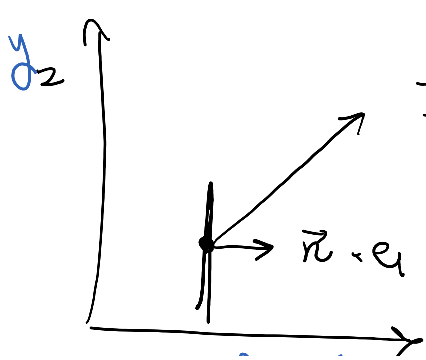
another proof in TAM 551



This is Newton's action-reaction formula

I told you that there is a stress tensor that maps normal vector to traction, but didn't prove it before.

Balance law (linear momentum) $\longrightarrow$ there exists a stress tensor (Cauchy stress tensor) that maps $\vec n_y \longrightarrow \vec t_y$

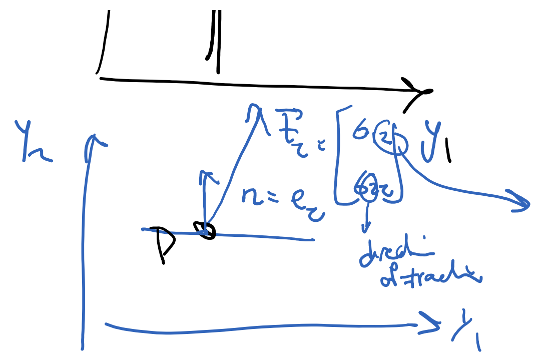


$$\vec t_1 = \begin{bmatrix} \sigma_{11} \\ \sigma_{21} \end{bmatrix}$$

traction component

normal direction

component

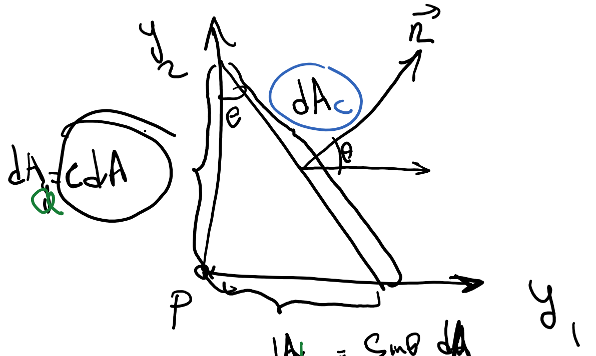


normal vector direction

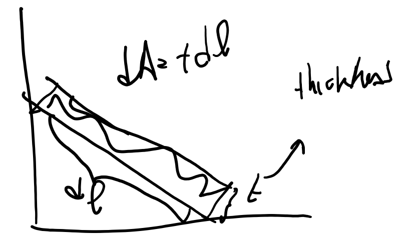
$\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}$ are unambiguously determined

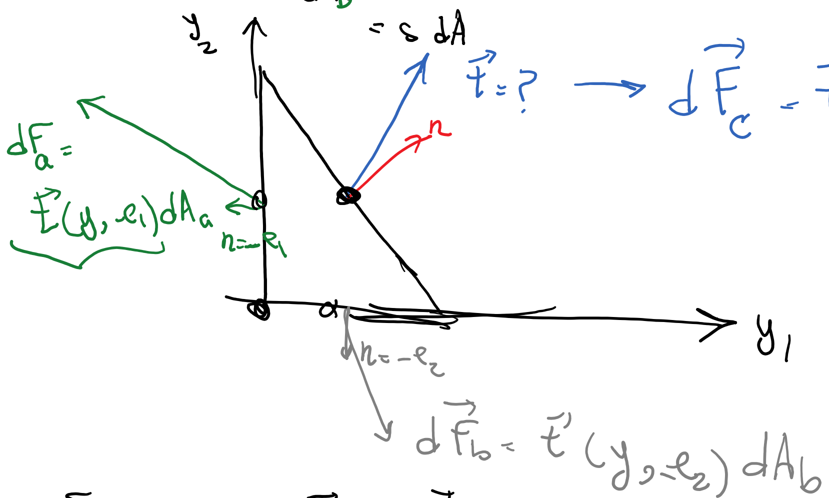
surface differential

dA



$dA_b = \sin\theta\, dA$
$= s\, dA$





$$\Sigma F = 0 \rightarrow d\vec{F}_a + d\vec{F}_b + d\vec{F}_c = 0 \rightarrow t(y, -e_1)(c\,dA)$$

$$+\ t(y, -e_2)(s\,dA) + t(y, \vec{n})(dA) = 0$$

$$\underbrace{\vec{t}(y, -e_1)}\, c + t(y, -e_2) \cdot s + t(y, \vec{n}) = 0$$

(*) $\rightarrow$ $-\vec{t}(y, e_1)\, c - t(y, e_2)\, s + t(y, \vec{n}) = 0$

Action-reaction

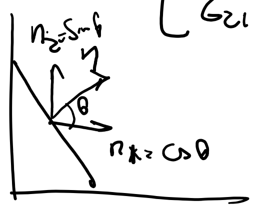
Recall I defined components of $t(y, e_1) := \begin{bmatrix} \sigma_{11} \\ \sigma_{21} \end{bmatrix} \rightarrow$

$$t(y,e_1) = \begin{bmatrix} \sigma_{11} \\ \sigma_{21} \end{bmatrix}$$

$$t(y,e_2) = \begin{bmatrix} \sigma_{12} \\ \sigma_{22} \end{bmatrix}$$

$$\vec{t}(y,n) = \underbrace{\begin{bmatrix} \sigma_{11} \\ \sigma_{21} \end{bmatrix} c}_{t(y,e_1)c} + \begin{bmatrix} \sigma_{12} \\ \sigma_{22} \end{bmatrix} s =$$

$$\vec{t}(y,\vec{n}) = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \underbrace{\begin{bmatrix} c \\ s \end{bmatrix}}_{n}$$



$$\vec{t}(y,n) = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

vector $\quad$ 2nd order

$$\vec{t}(y,n) = \overset{=}{\sigma} \cdot \vec{n}$$

Cauchy stress
denoted by T in TAM 551

scalar $\quad$ vector

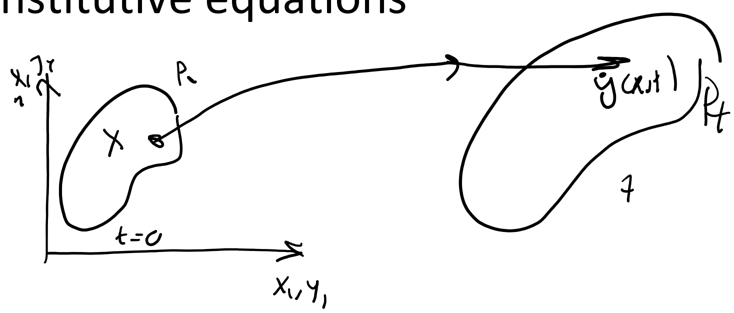
$$q_n = q \cdot n$$

We the same approach, if we know that there is a density of net heat flux through a surface we can right away prove that there is heat flux vector such that:

We proved stress tensor exists and $\quad \vec{t} = \overset{=}{\sigma} \cdot \vec{n}$

Constitutive equations

Elastic stress constitutive equation:



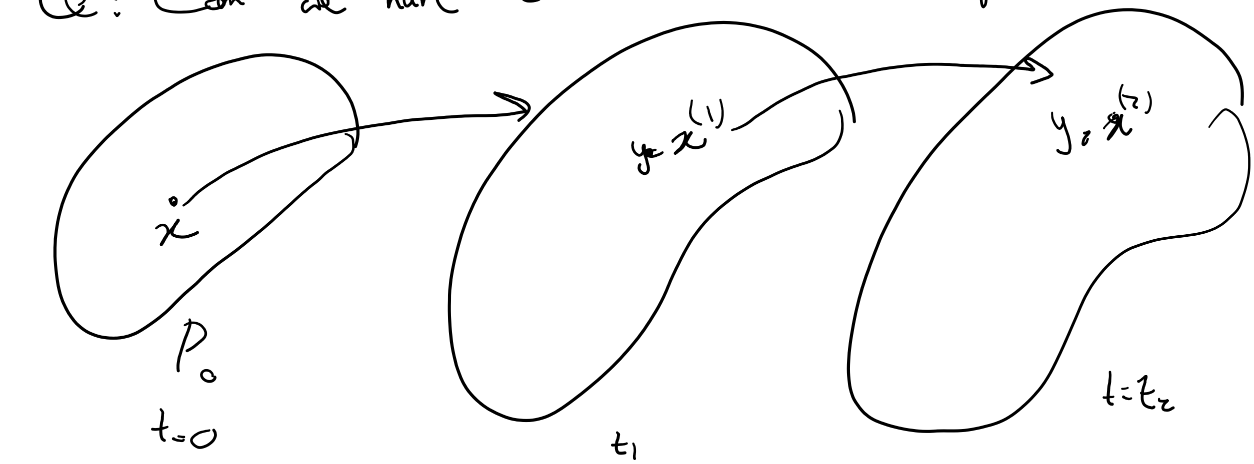
$$F(X,t) = \nabla_X y(X,t)$$

2nd order tensor.

$$\underset{\text{Cauchy stress}}{T}(X,t) = \underset{\text{Constitutive function}}{G}(\underset{\text{deformation gradient}}{F}, \underset{\text{dependency on position}}{X})$$

Cauchy's steps   Constitutive function

Q: Can we have different reference configurations.



$$\boxed{y = f(x,t)}$$

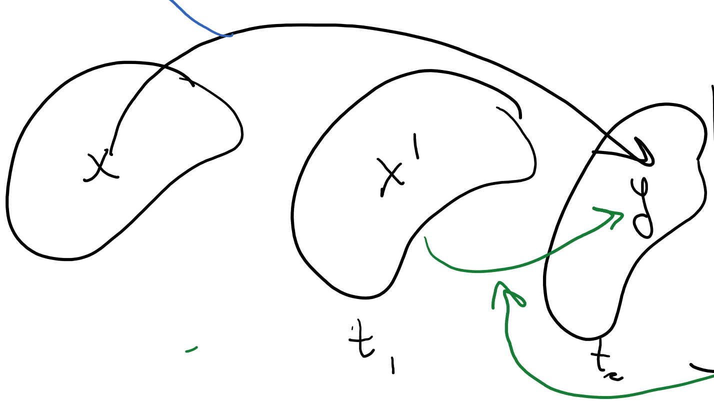
$x \longrightarrow y$

$$y = x^{(1)} = f(x, t_1)$$

$$\Rightarrow \quad x = f^{-1}(x^{(1)}, t_1)$$

$$y = f(x, t_2)$$

$$= f(f^{-1}(x^{(1)}, t_1), t_2 - t_1 + t_1)$$



$$y = f^{(2)}(x^{(1)}, t')$$

$$t' = t - t_1$$

$x^{(1)} \longrightarrow y$

Message:
If we have deformation map from one time to current time, we can form deformation map from any other reference time to now.

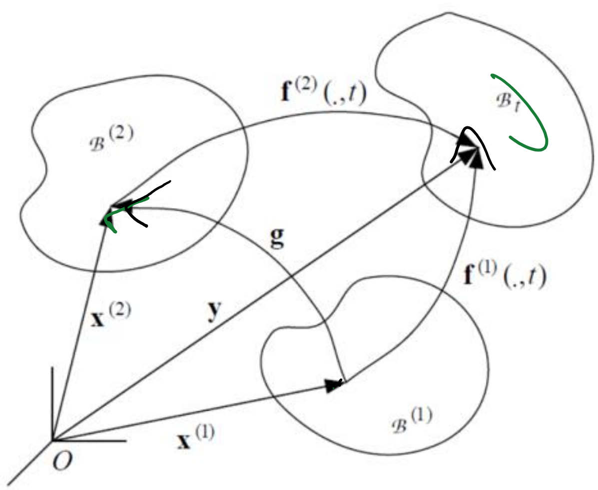
Remark 54 *It is commonly (but by no means universally) assumed that the reference configuration represents a* natural state of the body $\ni$ *the stress field vanishes (i.e., there is no* initial stress*). This assumption of course requires that*

$$\mathbf{G}(\mathbf{I}, \mathbf{x}) = \mathbf{0} \; \forall \quad \mathbf{x} \in \overset{0}{\mathcal{B}} .$$

Although this assumption is not always appropriate, we shall nonetheless adopt it for reasons of simplicity from here on.

We try to choose the reference configuration such that it's stress free

Now that we know how to relation deformation map between difference reference times, a question is how to relate the corresponding constitutive equations for stress tensor



Given $x^{(2)} = g(x^{(1)})$

we know the map between $x^{(1)}$ & $x^{(2)}$

Goal: Relate constitutive equations

from (1) → time t

& (2) → " "

we need $F^{(1)}$ & $F^{(2)}$ because const. eqn needs F

$$F^{(1)}_{ij} = \frac{\partial y_i}{\partial x^{(1)}_j} = \left[\frac{\partial y_i}{\partial x^{(2)}_k}\right] \frac{\partial x^{(2)}_k}{\partial x^{(1)}_j} = F^{(2)}_{ik} \frac{\partial g_k}{\partial x^{(1)}_j} = F^{(2)}_{ik} \nabla g_{kj}$$

$$\boxed{F^{(1)} = F^{(2)} \nabla g}$$

$$T = G^{(1)}(F^{(1)}, x^{(1)})$$

Cauchy stress

$$= G^{(2)}(F^{(2)}, x^{(2)})$$

$$T = G^{(1)}(F^{(2)} \nabla g, g^{-1}(x^{(2)})) = G^{(2)}(F^{(2)}, x^{(2)})$$

if $G^{(1)}$ & $g(x^{(1)} \to x^{(2)})$ are known $\Longrightarrow$ $G^{(2)}$ will be determined as

$$G^{(2)}(F^{(2)}, x^{(2)}) = G^{(1)}(F^{(2)} \nabla g, g^{-1}(x^{(2)}))$$

if $G^{(1)}$ & $g\left(X^{(1)} \to X^{(2)}\right)$ are known $\Longrightarrow$ $G^{(2)}$ will be determined as

$$G^{(2)}\left(F^{(2)}, X^{(2)}\right) = G^{(1)}\left(F^{(2)} \nabla g \, , \, g^{-1}\left(X^{(2)}\right)\right)$$