CM2021/11/23

Tuesday, November 23, 2021 4:29 PM

Frame-Invariance (objectivity)

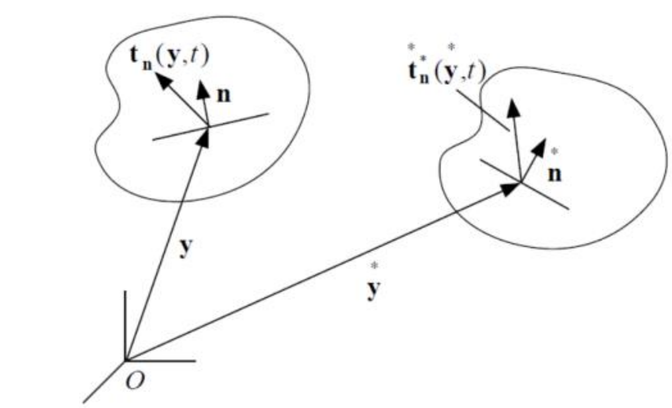


Figure 4.2: Surface tractions from equivalent motions

4.3 Principle of Material Frame-Indifference

This section explores the notion that material response is invariant under (indifferent to) superposed rigid motions and shifts in the origin of the time scale. Only invariance under superposed rigid motion is relavent in the context of elasticity theory which does not include memory effects. We begin with the notion of equivalent motions.

Definition 110 *Two motions of a body,* $\{\mathbf{f}(\cdot,t)\}$ *and* $\left\{\overset{*}{\mathbf{f}}(\cdot,t)\right\}$, *are equivalent w.r.t. material response if they differ by a rigid deformation for each* $t \in [t_0,\infty)$; *i.e.,* $\exists$ *functions* $\mathbf{c} : [t_0,\infty) \to \mathcal{V}$ *and* $\mathbf{Q} : [t_0,\infty) \to \mathrm{Orth}\,\mathcal{V}^+ \ni$

$$\overset{*}{\mathbf{f}}(\mathbf{x},t) = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{f}(\mathbf{x},t) \quad \forall\ (\mathbf{x},t) \in \overset{0}{\mathcal{B}} \times [t_0,\infty).$$

Next we compare the stress vectors of an elastic body associated with a pair of equivalent motions, $\{\mathbf{f}(\cdot,t)\}$ and $\left\{\overset{*}{\mathbf{f}}(\cdot,t)\right\}$. For the motion $\{\mathbf{f}(\cdot,t)\}$, consider a plane through point $\mathbf{y} \in \mathcal{B}_t$ with unit normal $\mathbf{n}$ and stress vector $\mathbf{t_n}(\mathbf{y},t)$. The corresponding items for the equivalent motion $\left\{\overset{*}{\mathbf{f}}(\cdot,t)\right\}$ are denoted $\overset{*}{\mathbf{y}}$, $\overset{*}{\mathbf{n}}$ and $\overset{*}{\mathbf{t}}_{\overset{*}{\mathbf{n}}}(\overset{*}{\mathbf{y}},t)$.

sees the deformation D as $y = f(x,t)$

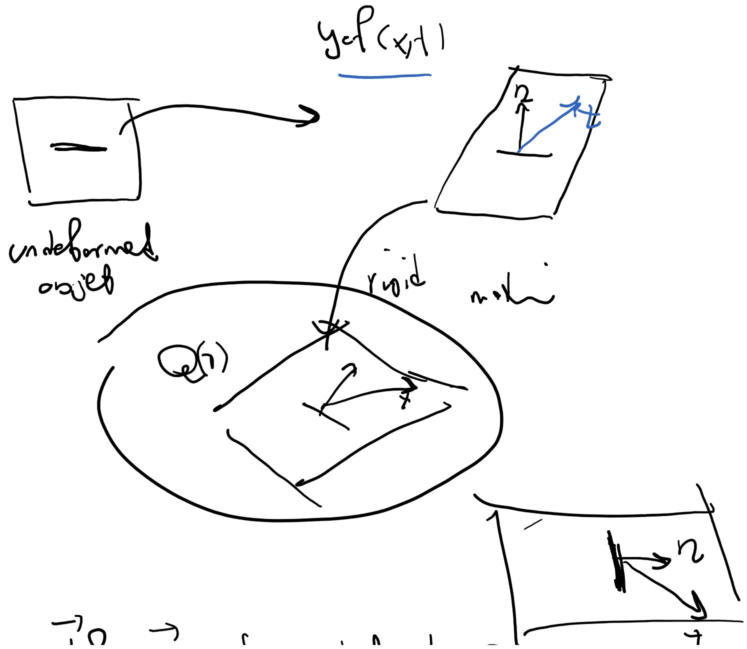
if the board have a rigid motion

$c(t)$ translation

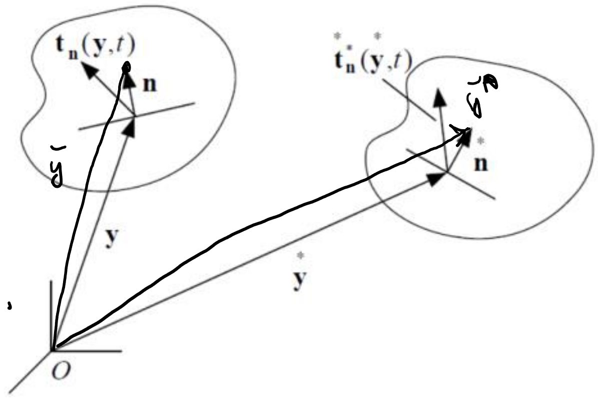
$Q(t)$ rotation

deformation of D is $\overset{*}{y} = c(t) + Q(t)\, y(x,t)$

we want to relate normal vector, traction vector, constitutive eqn between two observers



$\vec{t}$ & $\vec{n}$ also rotate by $Q(t)$



$\breve{y} = y + \vec{n}$

new normal will be aligned with $\breve{y}^* - y^*$

$$= \left(c(t) + Q(t)\breve{y}\right) - \left(c(t) + Q(t)\,y\right)$$

$$= Q(t)\,(\underbrace{\breve{y} - y}_{n}) = Q(t)\,n$$

n^* = normal vector along $\breve{y}^* - y^*$

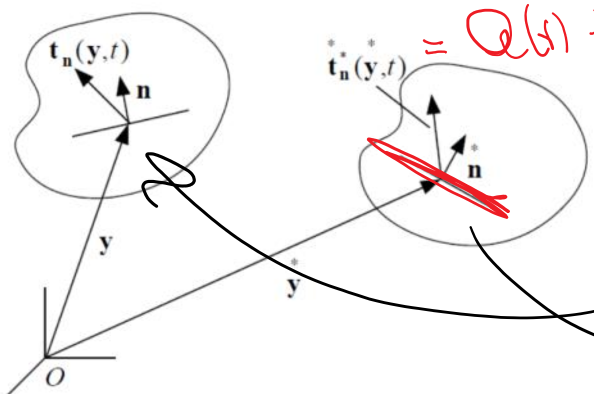
not $Q(t)n$ is size one. Why

$$Q(t)n \cdot Q(t) \cdot n \quad (\underbrace{Q^T Q}_{I}\, n) \cdot n = n \cdot n = 1$$

$$\boxed{n^* = Q(t)\,n}$$

Objectivity says

$$\underset{\text{acting on } n^*}{t^*} = Q\,\underset{\text{acting on } n}{t}$$



traction = stress $\cdot$ normal

$t_n = T\,n$

Cauchy stress

$t^*_{n^*} = T^*\,n^*$

Figure 4.2: Surface tractions from equivalent motions

$$\left\{ t_n^* = T^* n^* \right.$$

$$\left\{ \begin{array}{l} t_n^* = Q\, t_n \\ n^* = Q\, n \end{array} \right.$$

$$Q\, t_n = T^* Q\, n$$

$$t_n = T n \quad (\mathrm{D})$$

$$\rightarrow \quad Q\, T\, n = T^* Q\, n$$

vector n is arbitrary $\rightarrow$

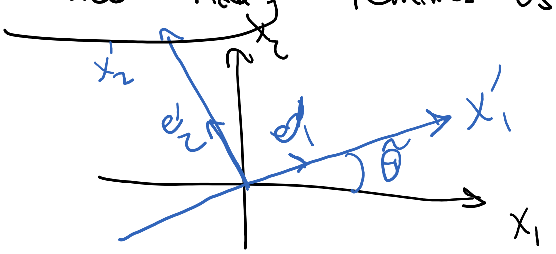
$$QT = T^* Q \quad \rightarrow \quad \boxed{T^* = Q\, T\, Q^t}$$

stress tensor

stress without outside rotation

for the observer with rigid rotation Q

Side note: reminds us of coordinate transformation rule



$$\tilde{Q} = \begin{bmatrix} e'_1 \\ \hline e'_2 \end{bmatrix} = \begin{bmatrix} \cos\tilde\theta & \sin\tilde\theta \\ -\sin\tilde\theta & \cos\tilde\theta \end{bmatrix}$$

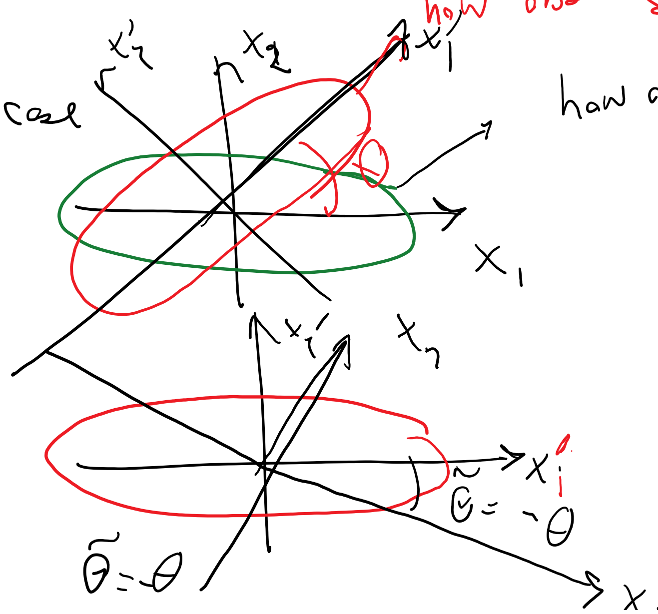
$$[T]' = \tilde{Q}\, [T]\, Q^t$$

* Object does not rotate
* Coordinate rotates

Here we have the opposite case
* Object rotates (by Q)
* Coordinate does not rotate

how observer 2 sees it

how observer 1 sees it



$$\tilde{Q} = \begin{bmatrix} \cos\tilde\theta & \sin\tilde\theta \\ -\sin\tilde\theta & \cos\tilde\theta \end{bmatrix}$$

$\tilde\theta = -\theta$

$$\tilde{Q} = \begin{bmatrix} \ldots & \ldots \\ -\sin\tilde{\theta} & \cos\tilde{\theta} \end{bmatrix} \qquad \tilde{\theta} = -\theta$$

$$= \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{actual rotation tensor } Q \text{ by angle } \theta} \qquad T^* = \tilde{Q}\, T\, \tilde{Q}^t$$

I resume with

$$T^* = Q T Q^t$$

Constitutive equation maps F from time 0 to time t to stress @ time t.

Observer 1 : $F = \nabla y_{/X}$ corresponding stress $T = G(F)$

" 2 $F^* = \nabla y^*_{/X}$ " " $T^* = G(F^*)$ $\downarrow$ QTQ^t, $\downarrow$ QF

$y^* = c + Qy$

$$F^*_{ij} = \frac{\partial y^*_i}{\partial x_j} = \frac{\partial\left(c_i(t) + Q(t)_{ik}\, y_k\right)}{\partial x_j}$$

$$= Q(t)_{ik} \frac{\partial y_k}{\partial x_j} = Q(t)_{ik} F_{kj} \to \boxed{F^* = QF}$$

basically

$$\left.\begin{aligned} T &= G(F) \\ Q T Q^t &= G(QF) \end{aligned}\right\}$$

$\forall Q$ rotation $\boxed{Q\, G(F)\, Q^t = G(QF)}$

$$F \xrightarrow{G} T$$

Objectivity for constitutive eqn

rotation

Objectivity for constitutive eqn puts some restrictions on the form of G

$F = RU$ polar decomposition

$$G(Q(RU)) = Q\,G(F)\,Q^t$$

$$G((QR)U) = Q\,G(F)\,Q^t$$

choose $Q = R^t$

$$G(U) = G((R^tR)U) = (R^t)\,G(F)\,(R^t)^t \longrightarrow$$

$$G(U) = R^t\,G(F)\,R \longrightarrow$$

(i) $$\boxed{G(F) = R\,G(U)\,R^t}$$

G is really a function of U pre & post multiplied by R

$F = RU \rightarrow R = FU^{-1}$

$$\rightarrow G(F) = \underbrace{FU^{-1}}_{R}\,G(U)\,\underbrace{(FU^{-1})^t}_{R} = F\underbrace{\left(U^{-1}G(U)\,U^{-t}\right)}_{\hat{G}(U) = U^{-1}G(U)\,U^{-t}}F^t$$

(ii) $$\boxed{G(F) = F\,\hat{G}(\underbrace{U}_{\sqrt{C}})\,F^t}$$

$U^2 = C = F^tF$

$$\bar{G}(C) = \hat{G}(\sqrt{C}) = \hat{G}(U)$$

(iii) $$\boxed{G(F) = F\,\bar{G}(C)\,F^t}$$

basically T should be written as a function of C not F

Theorem 173 *If the elastic constituitive equation*

$$\tilde{\mathbf{T}}(\mathbf{x},t) = \mathbf{G}(\mathbf{F}(\mathbf{x},t),\mathbf{x}) \tag{4.1}$$

is consistent with the Principle of Material Frame-Indifference, then it can be written in any of the following reduced forms*:*

$$\begin{aligned}
\tilde{\mathbf{T}}(\mathbf{x},t) &= \mathbf{R}(\mathbf{x},t)\mathbf{G}(\mathbf{U}(\mathbf{x},t),\mathbf{x})\mathbf{R}^t(\mathbf{x},t); && (4.2)\\
\tilde{\mathbf{T}}(\mathbf{x},t) &= \mathbf{F}(\mathbf{x},t)\hat{\mathbf{G}}(\mathbf{U}(\mathbf{x},t),\mathbf{x})\mathbf{F}^t(\mathbf{x},t); && (4.3)\\
\tilde{\mathbf{T}}(\mathbf{x},t) &= \mathbf{F}(\mathbf{x},t)\bar{\mathbf{G}}(\mathbf{C}(\mathbf{x},t),\mathbf{x})\mathbf{F}^t(\mathbf{x},t); && (4.4)
\end{aligned}$$

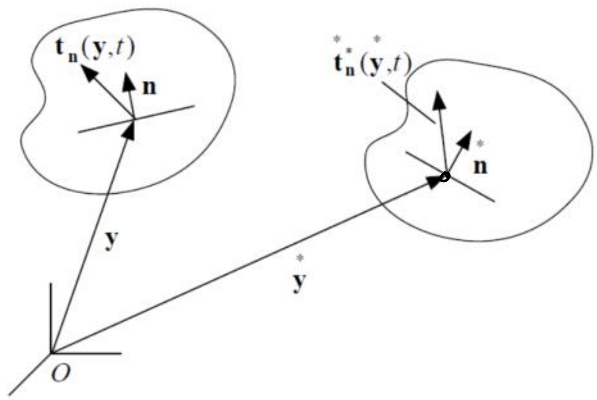
best form

2nd example of objectivity

Hyperelastic material is a material for which the internal energy density is a function of F

$e = \mathcal{E}(F)$

we'll show $\mathcal{E}$ is a function of C



$t_n^* = Q\, t_n$

$n^* = Q\, n$

$e^* = e$

$\mathcal{E}(F^*) = \mathcal{E}(F)$

$F^* = QF$

$\to \mathcal{E}(QF) = \mathcal{E}(F)$, $F \to RU$

$\mathcal{E}(QRU) = \mathcal{E}(F)$

choose $Q = R^T$

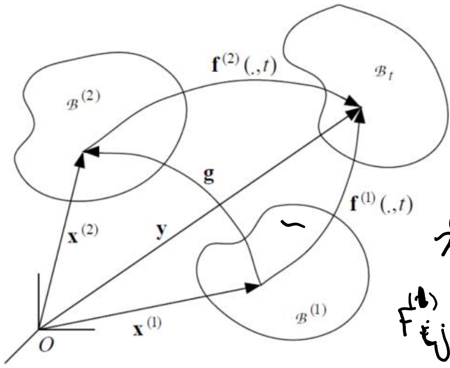
$\mathcal{E}(F) = \mathcal{E}(U) = \mathcal{E}(\sqrt{C}) = \bar{\mathcal{E}}(C)$

Summary
Objectivity restricts the form of constitutive equations:

	Starting point		Objectivity satisfied
General elastic	$T = G(F)$	$\longrightarrow$	$T = F\,\bar{G}(C)\,F^t$
Hyperelastic (more restricted space than elastic)	$e = \mathcal{E}(F)$	$\longrightarrow$	$e = \bar{\mathcal{E}}(C)$

How can we further simplify a constitutive equation using symmetry groups

Recall relating constitutive eqns from 2 different initial configurations (1) and 1 symmetry groups



Recall relating constitutive eqns from 2 different initial configurations (1), (2)

$$T = G^{(1)}(F^{(1)}, x^{(1)}) = G^{(2)}(F^{(2)}, x^{(2)})$$

$\nabla_y/x^{(1)}$ $\nabla_y/x^{(2)}$

$x^{(2)} = g(x^{(1)})$

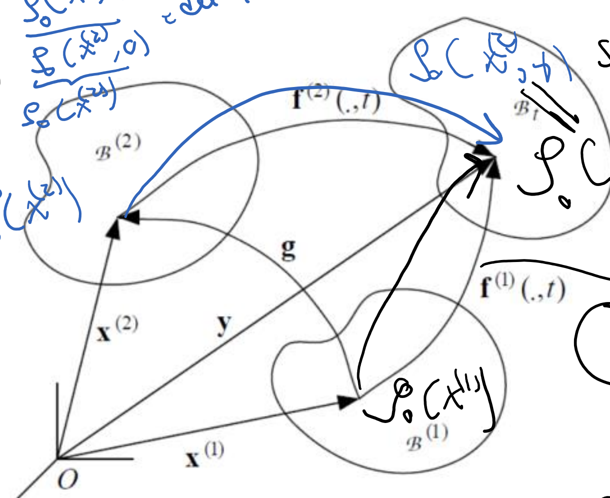
$$F_{ij}^{(1)} = \frac{\partial y_i}{\partial x_j^{(1)}} = \frac{\partial y_i}{\partial x_k^{(2)}} \frac{\partial x_k^{(2)}}{\partial x_j^{(1)}} = F_{ik}^{(2)} \nabla g_{kj}$$

$$\boxed{F^{(1)} = F^{(2)} \nabla g}$$

$$\boxed{G^{(1)}(F^{(2)}\nabla g, g^{-1}(x^{(2)})) = G^{(2)}(F^{(2)}, x^{(2)})}$$

$G^{(1)}$ known $G^{(2)}$ is obtained & vice versa

(2) $\frac{\rho_0(x^{(2)},t)}{\underbrace{\rho_0(x^{(2)},0)}_{\rho_0(x^{(2)})}} = \det F^{(2)}$



$\rho_0(x^{(2)}, t)$ Same density $\rho_0(x^{(1)}, t)$

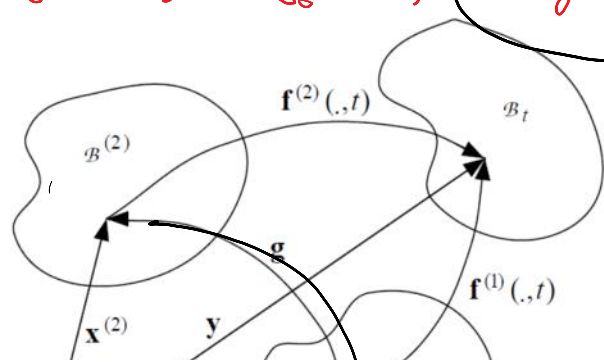
$\rho_0(x^{(2)})$

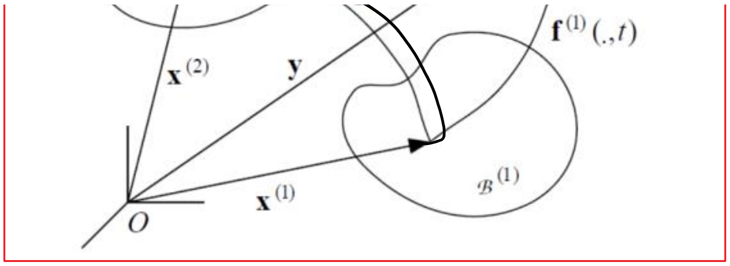
$\rho_0(x^{(1)})$

(1) $\frac{\rho_0(x^{(1)},t)}{\underbrace{\rho_0(x^{(1)},0)}_{\rho_0(x^{(1)})}} = \det F^{(1)} = \det(F^{(2)}\nabla g) = \det F^{(2)} \det \nabla g$

divide (1) by (2) & note $\rho_0(x^{(1)},t) = \rho_0(x^{(2)},t) \Rightarrow \frac{\rho_0(x^{(2)})}{\rho_0(x^{(1)})} = \det \nabla g$

$$\rho_0(x^{(2)}) = \rho_0(x^{(1)}) \det \nabla g$$





if $\det \nabla g = 1$ density is preserved between state 1 & 2

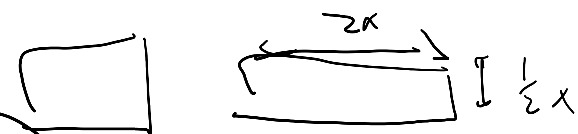
In terms of considering symmetry groups we only restrict ourselves to g's for which

$H = \nabla g$ is is unimodal $\boxed{\det H = 1}$

Examples

state 1

$$H = \nabla g = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

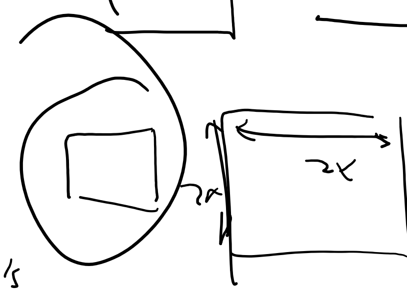


OK.

$$H = \nabla g = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$\det H \neq 1$

we won't consider these H's



(3) $\nabla g = Q$