CM2021/11/30

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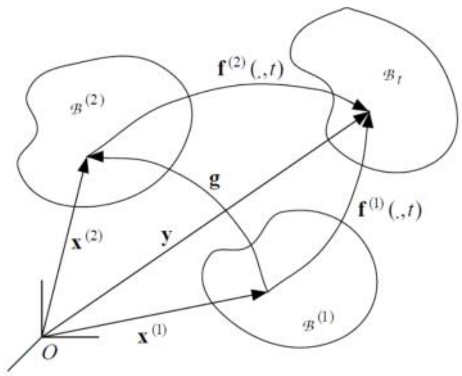
From last time

190 CHAPTER 4. ELASTIC RESPONSE



$$G^{(1)}(F\nabla g, g^{-1}(x^{(2)})) = G^{(2)}(F^{(2)}, x^{(2)})$$

$$\rho_0(x^{(2)}) = \rho_0(x^{(1)}) \det \nabla g$$

We restrict ourselves to maps for which density does not change from state 1 to 2

work with g's for which $H := \nabla g$ is unimodal

$$\det H = 1$$

basically density in states ① & ② will be one identical

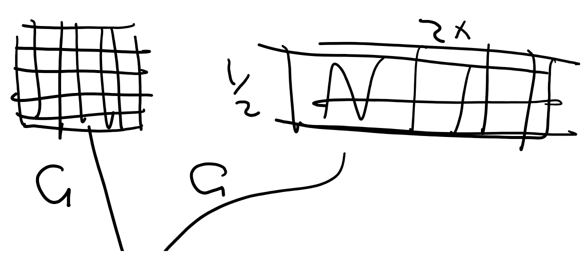
if for stress constitutive equation states ① & ② result in the same function expression we say $H = \nabla g$ belong to stress conv. eqn. symmetry group:

$$G^{(2)}(F) = G^{(1)}(F H)$$ ∇g

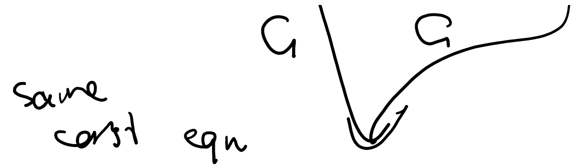
iff H in sym group $G^{(1)} = G^{(2)} = G$

$$\boxed{\begin{array}{l} \det H = 1, \quad G(FH, x) = G(F, x) \\ \text{H belongs to symmetry group} \\ \text{of stress function } G \end{array}}$$

Example: $H = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$



same



inviscid elastic fluid all unimodular H belong to G sym. group

$$T = G(F) = \bar{G}(\underbrace{\det F}_{J}) = \bar{G}\left(\frac{\rho}{\rho_0}\right) = \bar{\bar{G}}(\rho)$$

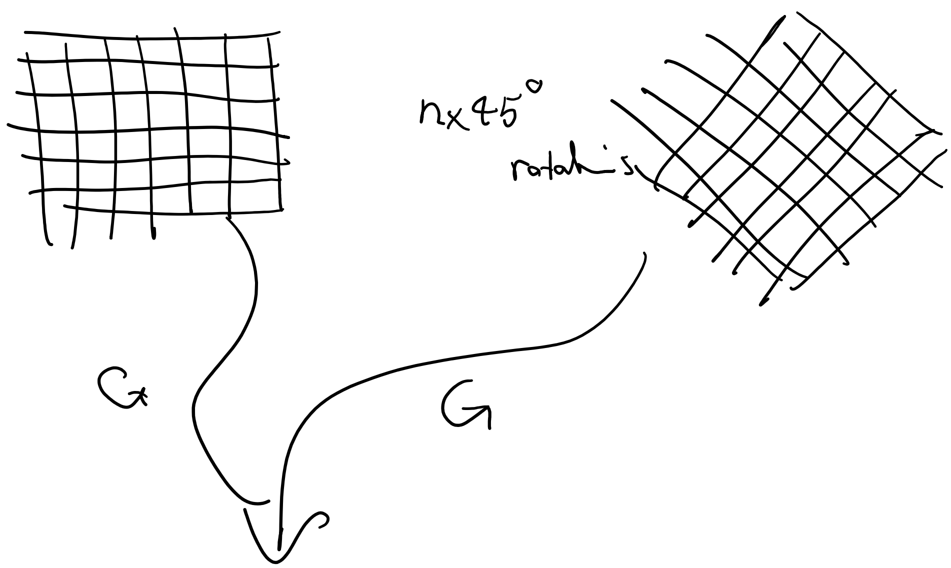
objectivity
a function of C ...

we can show it will be like

$$T = -\underbrace{p}_{\text{pressure}}(\rho, \theta)\, I$$

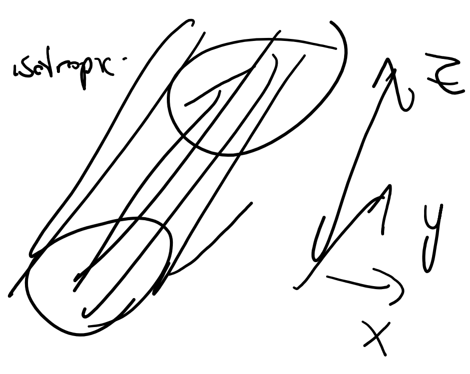
← temperature

Other example:



$H = Rot(n \times 45°)$ $\qquad G(F) = G(FH)$

transverse isotropic



isotropic in xy plane

$H = Rot_z(\theta)$ belongs to sym group

HW orthotropic $\mathrm{Rot}(n \times 90) \in$ sym. group

isotropic $\mathrm{Rot}(\theta)$ $0 \leq \theta \leq 360$ 2D

Voigt stiffness
$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ & C_{22} & C_{23} \\ & & C_{33} \end{bmatrix}_{3\times 3}$$
$C_{11} = C_{22}$ $C_{12} = C_{23} = C_{13} = 0$

$C_{33} = \dfrac{C_{11} - C_{12}}{2}$

3D elasticity isotropic material

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

Lame's parameter

Solid Mechanics, isotropic material is characterized by 2 parameters $((E,\nu), (\lambda,\mu), \ldots)$

Definition 112 (Noll, 1958) *Given an elastic body and a reference configuration that corresponds to the region* $\overset{0}{\mathcal{B}}$, *the* material symmetry group *at the material point identified by* $\mathbf{x}$ *in the reference configuration is the set*

$$\underset{\mathbf{x}}{\mathrm{Msg}} = \{\mathbf{H} \in \mathrm{Unim}\ \mathcal{V}^+ : \mathbf{G}(\mathbf{FH}, \mathbf{x}) = \mathbf{G}(\mathbf{F}, \mathbf{x})\ \forall\ \mathbf{F} \in \mathrm{Lin}\ \mathcal{V}^+\}.$$

Again, it should be emphasized that the material symmetry group is characterized by tensors $\mathbf{H}$ that correspond to the gradients at $\mathbf{x}$ of deformations — not the deformations themselves. This is because the mass density and the elastic response function in the second reference configuration depend only on the gradient of the connecting deformation. Also, note that $\mathbf{H} \in \mathrm{Msg}_{\mathbf{x}}$ is not a tensor field, but rather the value of a tensor field at $\mathbf{x}$.

The following theorem presents a property of all orthogonal elements of $\mathrm{Msg}_{\mathbf{x}}$ that derives from the Principle of Material Frame-Indifference.

What about H's that correspond to rotations, can we simply symmetry relation?

Theorem 174 is:

Let $Q \in \underbrace{\mathrm{Orth}^+ \mathcal{V}}_{\text{rotation}}$ then $Q \in \mathrm{Msg}_x$ iff

$$\boxed{\forall F \in Lin^+ \quad G(QFQ^t) = Q\,G(F)\,Q^t} \quad (*)$$

$Q \in Msq$ ① $G(F\overset{R}{Q}) = G(F)$

objective ② $G(QF) = Q\,G(F)\,Q^t$

use $F' = QFQ^t$ in ①

$$G((QFQ^t)Q) = G(QFQ^t)$$

$$G(QF(\underbrace{Q^tQ}_{I})) = \;=$$

$$G(QF) = G(QFQ^t)$$

we use ② to get

$$Q\,G(F)\,Q^t = G(QFQ^t)$$

What is elasticity 4th order tensor and what's Voigt notation?

(A) Cauchy stress

$$T = G(F) = \underbrace{F\,\bar{G}(C)\,F^t}_{\text{objectivity}}$$

"objective constitutive equation for finite elasticity" det H is not necessarily small

$H = F - I$

want to show in infinitesimal theory

(B) PK-I

$$S = \mathbb{C}\,E$$

infinitesimal strain

$E = \frac{1}{2}(H + H^T)$ $\quad \det H = \ll 1$

material is stress free at initial state

4th order elasticity tensor

Proof

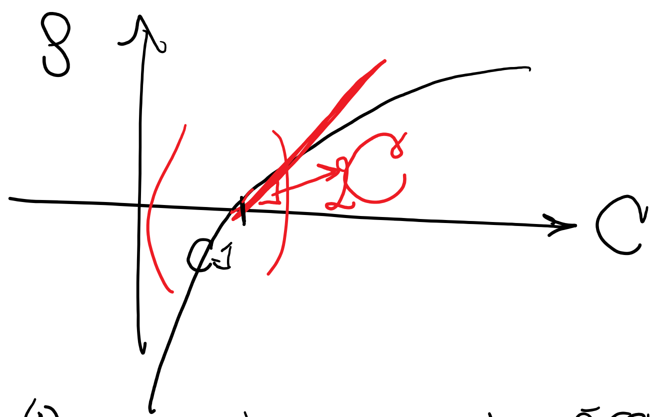
$$T = F\,\bar{G}(C)\,F^t$$

$$S = J\,T\,F^{-t} = J(F\,\bar{G}(C)\,F^t)\,F^{-t} \rightarrow \boxed{S = J\,F\,\bar{G}(C)}$$

①

$$\ldots = \sigma(F\,\bar{G}(C))\,F^{-1}F \rightarrow \boxed{S = \partial\,F\,\bar{G}(C)}$$ (1)

$C = F^tF$ at time zero $y = X$ $F = \nabla_X y = I$ so is $C = I$



Recall

$C = F^tF = (I+H)^t(I+H)$

$I + 2\frac{H+H^t}{2} + H^tH$

$$\boxed{C - I = 2E + O(\varepsilon^2)}$$

$= O(\varepsilon)$

in (1) I want to expand $\bar{G}(C)$

$$\bar{G}(C) = \bar{G}(\underbrace{C - I}_{\text{small number for infinidesimal elasticity}} + I) = \bar{G}(I) + \frac{\partial \bar{G}}{\partial C}\Big|_{C=I}(C - I) + HOT$$

small number for infinidesimal elasticity

$= O(\varepsilon)$

Taylor's expansion

like writing $g(1+\varepsilon) = g(1) + \varepsilon g' + O(\varepsilon^2)$

Why $\bar{G}(C=I) = 0$ (why when $F = I$, $T = 0$)

$y = X$

assumption that at initial state stress is zero (stress free initial configuration)

$$\bar{G}(C) = \frac{\partial \bar{G}}{\partial C}(C=I)\,(C - I) + O(\varepsilon^2)$$

4th order elasticity tensor is defined as

$$\mathcal{C} = 2\frac{\partial \bar{G}(C)}{\partial C}$$

4th order elasticity stiffness

Green strain

$$\mathcal{C}_{ijkl} = 2\left(\frac{\partial \bar{G}_{ij}(C)}{\partial C_{kl}}\right)$$

$$\mathbb{C}_{ijkl} = 2\left(\frac{\partial \ldots_{ij}}{\partial C_{kl}}\right)$$

$$\bar{G}(C) = \tfrac{1}{2}\,\mathbb{C}\,(C - I) + O(\varepsilon^2)$$

$$(C - I) = 2E + O(\varepsilon^2)$$

$$\bar{G}(C) = \tfrac{1}{2}\,\mathbb{C}\,(2E + O(\varepsilon^2)) + O(\varepsilon^2)$$

$$\boxed{\bar{G}(C) = \mathbb{C}\,E + O(\varepsilon^2)} \quad (2)$$

eqn (1)

$$\boxed{S = JF\,\bar{G}(C)}$$

$$S = JF\,(\mathbb{C}\,E) + O(\varepsilon^2)$$

$$[1 + \mathrm{trace}(E) + O(\varepsilon^2)]\;[I + \underbrace{H}_{O(\varepsilon)}]\;(\mathbb{C}\,E + O(\varepsilon^2))$$

lower order terms

$$S = 1.I\,\mathbb{C}\,E + \text{HOT}$$

$$S = \mathbb{C} E$$

$S \to$ PK-I

$\mathbb{C} \to$ 4th order elasticity tensor

$E \to \dfrac{H + H^T}{2}$

infinitesimal elasticity

$$S_{ij} = \mathbb{C}_{ijkl} E_{lk} = \mathbb{C}_{ijkl} E_{kl}$$

P_{ij}, T_{ij} are all the same in infinitesimal elasticity

$$S = J F \tilde{G}(C)$$

General finite elasticity

$$S = \mathbb{C} E$$

3D $S_{3\times3}$ $E_{3\times3}$ $\mathbb{C}_{3\times3\times3\times3}$ 81 components $\to$ 36 are independent

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix}$$

6 independent stresses

$$E = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix}$$

6

Voigt notation

$$s = \begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{12} \\ S_{23} \\ S_{31} \end{bmatrix} \qquad \text{eng strain } \gamma = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{12} \\ 2E_{23} \\ 2E_{31} \end{bmatrix}$$

$$\boxed{s = \tilde{C} \gamma}$$

$\tilde{C} \to$ 6x6 Voigt stiffness

6x6 Voigt stiffness

all 36 components can be independent for general elastic material

$C_{3\times3\times3\times3}$ we always have the following "minor" symmetries

$$\left[\begin{array}{l} C_{ijkl} = C_{jikl} \\ C_{ijkl} = C_{ijlk} \end{array}\right] \rightarrow \text{2 line proofs in TAM 551}$$

that's why we can can use Voigt notation.

Hyperelastic material:

It's an elastic material that whose internal energy density depends on F

objectivity

General elastic $T = \mathcal{G}(F) \longrightarrow T = F\,\bar{\mathcal{G}}(C)F^t$

hyperelastic material $e = \check{e}(F) \longrightarrow e = \bar{e}(C)$

internal energy density

known

what is $\bar{\mathcal{G}}$ for hyperelastic material

$$\boxed{\bar{\mathcal{G}}(C) = \frac{2\rho_0}{\sqrt{\det C}}\,\frac{\partial \bar{e}(C)}{\partial C}}$$

Very short proof:

I'll give you a derivation that shows

arbitrary

I'll give you a derivation that shows

sym $\underbrace{2\rho\, F \frac{\partial \bar{e}}{\partial C} F^t}$: $D = T : D$ → arbitrary sym. tensor

sym

$L = \text{grad } v_s \qquad L_{ij} = \frac{\partial v_i}{\partial y_j}$

$D = \frac{L + L^T}{2}$ stretching tensor (recall $F = R\,U$, U stretch tensor)

$W = \frac{L + L^T}{2}$ spin tensor

$$T = 2\rho\, F \frac{\partial \bar{e}}{\partial C} F^t$$

$$= F\left(2\frac{\rho_0}{J}\frac{\partial \bar{e}}{\partial C}\right)F^t = F\, \bar{G}(C)\, F^t$$

T from objectivity

$$\longrightarrow \quad \bar{G}(C) = \frac{2\rho_0}{\det F}\frac{\partial \bar{e}}{\partial C}$$

$$\boxed{\bar{G}(C) = \frac{2\rho_0}{\sqrt{\det C}}\frac{\partial \bar{e}}{\partial C}} \quad (1)$$

$\bar{e}$ energy function $\longrightarrow$ $\bar{G}$ stress constitutive eqn

I want to derive $\mathcal{C}_{ijkl}$ for hyperelastic materials

$$\bar{G}(C = I) = \frac{2\rho_0}{\sqrt{1}}\frac{\partial \bar{e}}{\partial C}(C = I) = 0$$

stress free initial condition

$$\rightarrow \boxed{\frac{\partial \bar{e}}{\partial C}(C = I) = 0} \quad (2)$$

Recall $\mathcal{C} = 2\left.\frac{\partial \bar{G}}{\partial C}\right|_{C=I} \rightarrow$

$$\partial C \,|\, C=I$$

$$\tilde{\mathcal{C}}_{ijkl} = 2 \frac{\partial \bar{G}_{ij}}{\partial C_{kl}} (C=I)$$

$$\bar{G}_{ij} = \frac{2\rho_0}{\sqrt{\det C}} \frac{\partial \bar{e}}{\partial C_{ij}}$$

$$\tilde{\mathcal{C}}_{ijkl} = \cancel{4}\rho_0 \frac{\partial \frac{1}{\sqrt{\det C}}}{\partial C_{kl}} \underbrace{\frac{\partial \bar{e}}{\partial C_{ij}} (C=I)}_{=0}$$ because of 2

$$+ \frac{4\rho_0}{\underbrace{\sqrt{\det C}}_{C=I}} \frac{\partial^2 e}{\partial C_{kl} \partial C_{ij}} (C=I)$$

$$\underbrace{\tilde{\mathcal{C}}_{ijkl} = 4\rho_0 \frac{\partial^2 \bar{e}}{\partial C_{kl} \partial C_{ij}}}$$

major symmetry $= 4\rho_0 \frac{\partial^2 \bar{e}}{\partial C_{ij} \partial C_{kl}} = \mathcal{C}_{klij}$

Only for hyperelastic

$$\begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{12} \\ S_{23} \\ S_{31} \end{bmatrix} = \tilde{\mathcal{C}}_{6\times 6} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{12} \\ 2E_{23} \\ 2E_{31} \end{bmatrix}$$

sym with 21 components