CM2023/08/23

Wednesday, August 23, 2023 9:37 AM

https://rezaabedi.com/teaching/continuum-mechanics/

Course assignments are mostly from
TAM551.pdf

Abeyaratne_Brief Review of Some Mathematical_ElasticSolids-Vol.1-Math.pdf
Abeyaratne_Continuum Mechanics_RCA_Vol_II.pdf

Grade breakdown:
Mostly HW assignments plus a term project

9 HW assignments (about 80%)

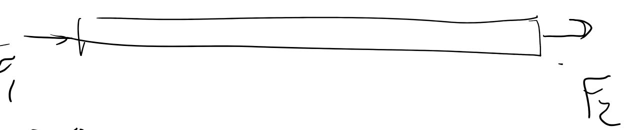
Term project 1: Includes (About equal weights are allocated to each part)

- 1) An up to 4 pages paper/proposal(including references if any) on a topic related to continuum mechanics. The format of the document is either that of a
  - **Research article** mostly focusing on introducing a topic of interest and presenting related results. Suggested sections are abstract, introduction, formulation, results (can present results from existing literature, doesn't need to be from your own research), conclusion.
  - **Research proposal** that basically introduces a problem, discusses current state of the art and research gaps, and finally proposes a new approach to address the mentioned research gaps. Suggested sections are (abstract), introduction (why this problem is important and what is the main contribution of the proposed work), background (state of the art and what are the existing gaps and challenges), objective (describing the goal and objectives of the research), research tasks (what is proposed to be done). Some optional sections are intellectual merits and broader impacts as often required in research proposals.
- 2) Presentation of the article on the "Presentation day". Each student will have about 15 minutes to present the material in the article (and related to it) to the entire class.

(i) kinematics (geometrical description of deformation);

(ii) basic balance laws; and

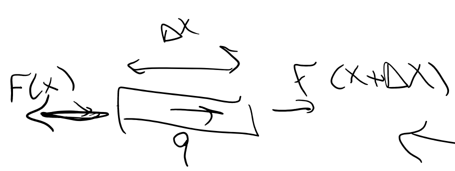
(iii) constitutive theories for both fluids and solids.

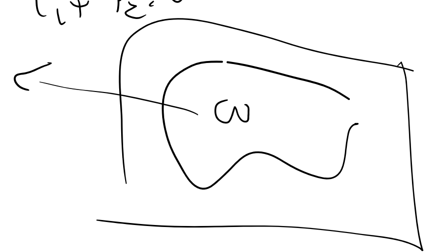


1 Balance laws

$\sum F = 0$ $\quad F_1 + F_2 = 0$

$\sum_{\omega} F = 0$





$\sum F = 0$

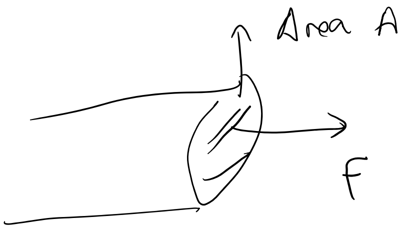
$F(x+\Delta x) - F(x) + q\,\Delta x = 0$

divide by Δx $\quad \frac{F(x+\Delta x) - F(x)}{\Delta x} = -q \quad$ let $\Delta x \to 0$

(eq1) $F'(x) = -q$ $\quad$ Differential equation

stress

$\sigma = \frac{F}{A} \Rightarrow F = A\sigma$ (eq2)

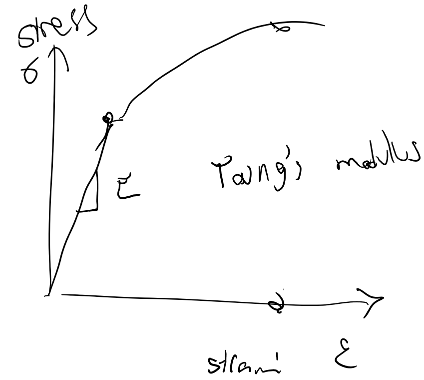


$\sigma = \dfrac{F}{A}$ $\Rightarrow$ $F = A\sigma$ (eq2)

intensity of force

(eq1) & (eq2) $\rightarrow$ $(A\sigma(x))' + q = 0$ (eq3)

② Constitutive equations



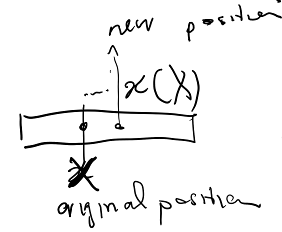
$\sigma = E\varepsilon$ (eq 4)

(eq3) & (eq4) $\Rightarrow$ (eq 5) $\left(A(x)\,E(x)\,\varepsilon(x)\right)' + q(x) = 0$

③ Kinematics

$u(x)$ displacement field $\rightarrow$ $\varepsilon(x) = \dfrac{du}{dx}$ (eq6)

$v(x) = u'(x)$



displacement

$u(X) = x(X) - X$

(eq6) & (eq5)

$$\Rightarrow \boxed{\frac{d}{dx}\left(A(x)\,E(x)\,\frac{du}{dx}\right) + q(x) = 0}$$

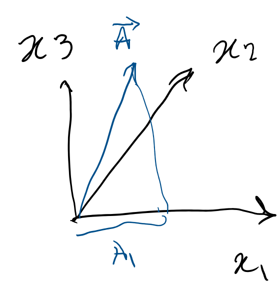
Final Differential equation

First section:
Mathematical background:
- Indicial notation
- Vector spaces
- Tensors

Indicial notation:

d: dimension

3D: $d = 3$



Consider a vector A

$A = (A_1, A_2, A_3)$

For this vector $A_1 > 0$, $A_2 > 0$, $A_3 > 0$

or $\underbrace{\forall i \in \{1,2,3\}}_{\text{what if we drop this?}}$ $A_i > 0$

shorthand

$\boxed{A_i > 0}$

Free indices appear once per expression and they imply that the index takes values from 1 to d (e.g., 1, 2, 3 in 3D)

$\Rightarrow$ $\boxed{A_i > 0}$

free index.

Free indices appear once per expression and they imply that the index takes values from 1 to d (e.g., 1, 2, 3 in 3D)

We deal with addition and subtractions of terms as follows:

$\forall i \in \{1,2,3\}$ $\quad \underline{a_i} + A_{ij} b_j + C_{i2} d_2 = \frac{d\, d_{ik}}{d x_k} + \cos(e_i) + \cancel{x_k} + 5 \quad \cancel{+ b_i b_i}$

i appears just once in all the terms in this expression

just 1 i is allowed

Other examples:
The following expression does not make sense

$\times \quad a_i + b_j > 0$ $\qquad$ I wanted to state $\left.\begin{matrix} a_1+b_1>0 \\ a_2+b_2>0 \\ a_3+b_3>0 \end{matrix}\right] \equiv a_i + b_i > 0$

free index

$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \quad \underset{3\times3}{B} = \ldots$

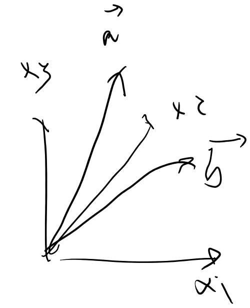
$C = A + B$ $\qquad \underset{3\times3}{\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ - & - & \\ - & & C_{33} \end{bmatrix}} = \begin{bmatrix} A_{11} & & A_{13} \\ & & \\ & & A_{33} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ & - & - \\ & & B_{33} \end{bmatrix}$

Much easier to write

$$C_{ij} = A_{ij} + B_{ij}$$

this is a shorthand for $\underbrace{\forall i \in \{1,2,3\}, j \in \{1,2,3\}}_{\text{we just don't write this}}$ $C_{ij} = A_{ij} + B_{ij}$

Examples of vector operations and products



$\vec{a} = \vec{b} \iff \forall i \quad a_i = b_i \quad (a_i = b_i)$

or $a_k = a_k$

$\underset{3\times1}{\vec{a}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad \underset{3\times1}{\vec{b}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$a^t b = [a_1 \quad a_2 \quad a_3]_{1\times3} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{3\times1} = a_1 b_1 + a_2 b_2 + a_3 b_3$ scalar

$$a^t b = [a_1 \ a_2 \ a_3]_{1\times 3} \begin{bmatrix} b_2 \\ b_3 \end{bmatrix}_{3\times 1} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

scalar

inner product

$$\boxed{\vec{a} \cdot \vec{b} = a^t b = \sum_{i=1}^{3} a_i b_i} \quad (1)$$

$$a b^t = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} [b_1 \ b_2 \ b_3] = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$$

(2) $$(\vec{a} \otimes \vec{b})_{3\times 3} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \quad \boxed{\forall i,j \in \{1,2,3\} \quad (a \otimes b)_{ij} = a_i b_j}$$

dyadic product

shorthand with indicial notation $(a \otimes b)_{ij} = a_i b_j$

(1) $$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{i=1}^{3} a_i b_i$$

index i appearing twice implies summation

2nd convention

repeated index implies summation

dummy index

Example $a \cdot b = a_i b_i$ (short hand for $\sum_{i=1}^{3} a_i b_i$)

$= a_k b_k$

$= a_\alpha b_\alpha$ Einstein summation convention

$$a_k b_k = \sum_{k=1}^{3} a_k b_k = \sum_{i=1}^{3} a_i b_i$$

$a_i \geq 0$
free index
$a_i + b_i > 5$

$a_i b_i \geq 0$
dummy index

More examples

$$\underset{3\times 3}{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & - & \\ & & C_{33} \end{bmatrix}$$

Meaning of $\cancel{\sum_{i=1}^{3}} C_{ii} \geq 0 \quad = \quad C_{11} + C_{22} + C_{33} \geq 0$

... to say each diagonal component is ≥ 0

what if we want to say each diagonal component is ≥ 0

$-\ C_{ij} > 0$ ✗ $C_{11}>0 \ldots \cancel{C_{33}>0}$ 9 eqns

$-\ C_{ii} > 0$ $C_{11}+C_{22}+C_{33}>0$ ✗

when summation convention is not implied

$C_{ii} > 0$ (no summation convention)

$C_{i\underline{i}} > 0$ → explain $\underline{i}$ or (i) mean no summation on i

$C_{i(i)} > 0$

$b = Aa$

vector matrix (capital letter)

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

free index

$b_1 = A_{11}a_1 + A_{12}a_2 + A_{13}a_3$ → dummy index

$b_2 = A_{21}a_1 + A_{22}a_2 + A_{23}a_3$

$b_3 = A_{31}a_1 + A_{32}a_2 + A_{33}a_3$

$\forall\, i \in \{1,2,3\}$ $\quad b_i = \sum_{j=1}^{3} A_{ij} b_j$

$b_i = A_{ij} b_j$

$\boxed{b_i = A_{ij}\, a_j} \equiv b = Aa$

$b = Aa$, $c = Bb$

write c in terms of a using indicial notation

$c = Bb$, $b = Aa$ $\qquad c = \underbrace{(BA)}_{\text{call this matrix } C} a$

How are components of $C = BA$ are related to components of A & B?

need to change this to j $\quad i \to j$

$b = Aa$: $b_i = A_{ij}\, a_j$ $\qquad$ ~~$b_j = A_{jj}\, a_j$~~ $(3\,j)$ ✗

$b = Aa \quad : \quad b_i = A_{ij} a_j \qquad \cancel{b_j = A_{jj} a_j} \quad (3j)$ X

$c = Bb \qquad \boxed{c_i = B_{ij} b_j}$ (3)

$b_i = A_{ik} a_k \qquad \left(b_i = \sum_{k=1}^{3} A_{ik} b_k\right)$

$i \to j$

now we can change $i \to j$

(4) $\boxed{b_j = A_{jk} a_k}$

(3),(4) $\Longrightarrow \quad c_i = B_{ij}(A_{jk} a_k)$

$c_i = (B_{ij} A_{jk}) a_k$

$c = \underbrace{(BA)}_{C} a \qquad c_i = C_{ik} a_k$

$\Longrightarrow C_{ik} = (BA)_{ik} = \underbrace{B_{ij} A_{jk}}_{\text{summati on } j}$

(5) $\boxed{(BA)_{ik} = B_{ij} \underbrace{A_{jk}}_{\text{summati}}}$

$$BA = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$(BA)_{23}$ (i = 2, k = 3) $= B_{21} A_{13} + B_{22} A_{23} + B_{23} A_{33} = \sum B_{2j} A_{j3}$