CM2023/08/28

Monday, August 28, 2023 9:38 AM

Eigenvalue problem and why often indicial notation does not work for eigen-value, eigen-vector expressions

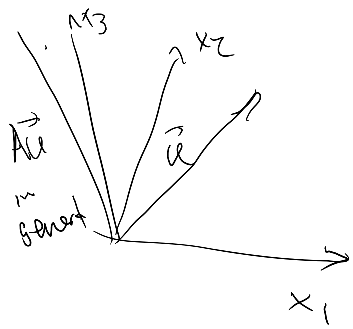
$$A u = \lambda u$$

u: eigenvector

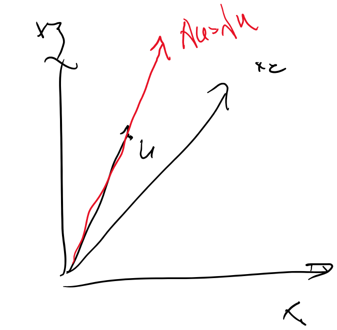
A: $n \times n$ square matrix

n: eigenvalues

λ: eigenvalue $\lambda_1, \dots, \lambda_n$



for eigenvectors



Back to $(d=3)$: 3×3 matrices

Assume A is diagonalizable (we have 3 independent eigenvectors) [if λ's are distinct → always the case]

we have 3 eigenvalues $\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}$ & corresponding 3 eigenvectors $u^{(1)}, u^{(2)}, u^{(3)}$

$$① \begin{cases} A u^{(1)} = \lambda^{(1)} u^{(1)} \\ A u^{(2)} = \lambda^{(2)} u^{(2)} \\ A u^{(3)} = \lambda^{(3)} u^{(3)} \end{cases}$$

$$\therefore \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_3^{(1)} \end{bmatrix} = \lambda^{(1)} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_3^{(1)} \end{bmatrix}$$

Can we write this expression using indicial notation?

$$\underbrace{A}_{\text{1 index } i \text{ here}} u^{(i)} = \underbrace{\lambda^{(i)} u^{(i)}}_{\text{2 indices } i \text{ here}}$$

it implies summation: $\lambda^{(1)} u^{(1)} + \lambda^{(2)} u^{(2)} + \lambda^{(3)} u^{(3)}$

Summation convention does not apply here

$$\boxed{A u^{(i)} = \lambda^{(i)} u^{(i)} \quad \text{no summation convention}}$$

① says

$$A \left[u^{(1)} \middle| u^{(2)} \middle| u^{(3)} \right] = \left[\lambda^{(1)} u^{(1)} \middle| \lambda^{(2)} u^{(2)} \middle| \lambda^{(3)} u^{(3)} \right]$$

$Au^{(1)} = \lambda^{(1)} u^{(1)}$

$$= \underbrace{\left[u^{(1)} \middle| u^{(2)} \middle| u^{(3)} \right]}_{\text{eigenvector matrix}} \underbrace{\begin{bmatrix} \lambda^{(1)} & & \\ & \lambda^{(2)} & \\ & & \lambda^{(3)} \end{bmatrix}}_{\text{eigenvalue matrix}}$$

diagonal matrix on the right → multiply columns

diagonal matrix on the left → multiply rows

eigenvector matrix: column i is eigen vector i

eigenvalue matrix: diagonal i is eigen vector i

column i is eigen vector i

diagonal i is eigen vector i

(2) $\boxed{AU = U\Lambda}$ $\Longrightarrow$ multiply b U^{-1} on the right

$AUU^{-1} = U\Lambda U^{-1}$ $\Longrightarrow$ (3) $\boxed{A = U\Lambda U^{-1}}$

diagonalizing a matrix

Side note: Why diagonalizing a matrix is so important?

$$A^2 = (U\Lambda U^{-1})(U\Lambda U^{-1}) = U\Lambda(\underbrace{U^{-1}U}_{I})\Lambda U^{-1} = U\Lambda^2U^{-1}, \quad \Lambda^2 = \begin{bmatrix} \lambda^{(1)2} & & \\ & \lambda^{(2)2} & \\ & & \lambda^{(3)2} \end{bmatrix}$$

$$A^n = U\Lambda^n U^{-1} \qquad \Lambda^n = \begin{bmatrix} \lambda^{(1)n} & & \\ & \lambda^{(2)n} & \\ & & \lambda^{(3)n} \end{bmatrix}$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

$$\exp(A) = U \begin{bmatrix} \exp(\lambda^{(1)}) & & \\ & \exp(\lambda^{(2)}) & \\ & & \exp(\lambda^{(3)}) \end{bmatrix} U^{-1} \qquad \exp(y) = e^y$$

(4) $$\boxed{f(A) = U \begin{bmatrix} f(\lambda^{(1)}) & & & \\ & f(\lambda^{(2)}) & & \\ & & \ddots & \\ & & & f(\lambda^{(n)}) \end{bmatrix} U^{-1}}$$

$$A^{-1} = U\Lambda^{-1}U^{-1}$$

Back to indicial notation problems with eigen problems

$A = U\Lambda U^{-1}$ if A is symmetric then $U^{-1} = U^t \rightarrow$ transpose

we'll show this later

(5) $\boxed{\text{for symmetric } A \quad A = U\Lambda U^t}$

$$A = \left[u^{(1)} \middle| u^{(2)} \middle| u^{(3)} \right] \begin{bmatrix} \lambda^{(1)} & & \\ & \lambda^{(2)} & \\ & & \lambda^{(3)} \end{bmatrix} \begin{bmatrix} u^{(1)} \\ \hline u^{(2)} \\ \hline u^{(3)} \end{bmatrix}$$

$$A = \lambda^{(1)} u^{(1)} \otimes u^{(1)} + \lambda^{(2)} u^{(2)} \otimes u^{(2)} + \lambda^{(3)} u^{(3)} \otimes u^{(3)}$$

$$A = \lambda^{(1)} u^{(1)} \otimes u^{(1)} + \lambda^{(2)} u^{(2)} \otimes u^{(2)} + \lambda^{(3)} u^{(3)} \otimes u^{(3)}$$

$\hookrightarrow$ dyadic product

(6) sym. A

$$\boxed{A = \sum_{i=1}^{3} \lambda^{(i)} u^{(i)} \otimes u^{(i)}}$$

$$a \otimes b = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$$

Can we use the summation convention?

$$A = \lambda^{(i)} u^{(i)} \otimes u^{(i)}$$

we cannot drop Σ here

"cannot use summation convention"

because there are 3 indices i on the RHS

$d=3$

Kronecker's delta

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

$$\delta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Identity matrix

So, delta is the identity matrix

Properties of δ:

1. $\delta_{i\underline{i}} = 1$ no summation convention.

2. $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$ (d in dimension d)

3. $\underbrace{\delta_{ij} b_j}_{\text{summation}} = \delta_{i1} b_1 + \delta_{i2} b_2 + \delta_{i3} b_3 = b_i$

e.g. $i=1$ $\underbrace{\delta_{11}}_{1} b_1 + \underbrace{\delta_{12}}_{0} b_2 + \underbrace{\delta_{13}}_{0} b_3 = b_1$

$i=2$ $\underbrace{\delta_{21}}_{0} b_1 + \underbrace{\delta_{22}}_{1} b_2 + \underbrace{\delta_{23}}_{0} b_3 = b_2$

$i=3$ b_3

(7)

$$\boxed{\delta_{ij} b_j = b_i}$$

$\delta_{ji} b_j =$

$$\boxed{a_i\, b_i = b_i}$$

$$\delta_{ji} C_{klip} = \delta_{ij} C_{kljp} = C_{klip}$$

$$\delta_{ij} C_{klji} = C_{klii} \qquad (= C_{kl11} + C_{kl22} + C_{kl33})$$

$c = Aa + 5a$ write $c = (\underset{\text{matrix}}{?})\, a$

$$c_i = (Aa + 5a)_i = \underline{(Aa)_i} + 5a_i$$

$$= A_{ij} a_j + 5a_i$$

not the same :(

trick to change a_i to a_j

note $a_i = \delta_{ij} a_j$

$$= A_{ij} a_j + 5\delta_{ij} a_j = (A_{ij} + 5\delta_{ij}) a_j$$

:)

$$c_i = (A_{ij} + 5\delta_{ij}) a_j$$

$$c = Ca: \quad c_i = C_{ij} a_j$$

$$\boxed{c = Ca \quad \text{where } C_{ij} = A_{ij} + 5\delta_{ij} \quad \text{or } C = A + 5I}$$

side note easier way $c = Aa + 5Ia = (A + 5I)a$

Permutation or Alternating symbol

$$\epsilon_{ijk}$$

$\{1, 2, 3\}$

$3 \times 3 \times 3 = 27$ components

e.g. ϵ_{111}

ϵ_{132}

examples

$\epsilon_{112} = \epsilon_{223}$

(8) Definition

$$\epsilon_{ijk} = \begin{cases} 0 & i=j \text{ or } j=k \text{ or } i=k \quad (21) \\ 1 & \text{positive permutation} \quad \epsilon_{123}=\epsilon_{231}=\epsilon_{312} \quad (3) \\ -1 & \text{negative permutation} \quad \epsilon_{132}=\epsilon_{321}=\epsilon_{213} \quad (3) \end{cases}$$

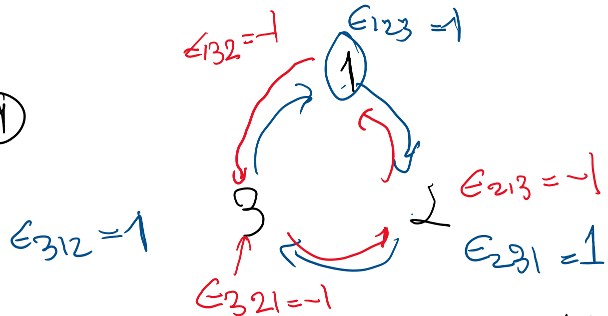
6 cases where i, j, k are distinct values from set $\{1,2,3\}$

possibilities (27)

examples $\epsilon_{112} = \epsilon_{223} = \epsilon_{313} = 0$

How to choose 1 or -1 for the remaining 6 cases $i \neq j$, $j \neq k$, $k \neq i$

Idea (1)



positive permutation $(+1)$

$\overset{\frown}{\epsilon_{123}} \rightarrow \epsilon_{213}$ 1 permutation

	starting value	# permutations			
ϵ_{123}	ϵ_{123}		0	1	$+1$
ϵ_{231}	ϵ_{123}	$\epsilon_{213} \rightarrow \epsilon_{231}$ even # permutations	2	1	
ϵ_{312}	ϵ_{123}	$\epsilon_{321} \rightarrow \epsilon_{312}$	2	1	
ϵ_{132}	ϵ_{123}	ϵ_{132}	1	-1	
ϵ_{321}	ϵ_{123}	ϵ_{321} odd # permutation	1	-1	
ϵ_{213}	ϵ_{123}	ϵ_{213}	1	-1	

idea 2: $\begin{cases} \text{even permutation} \rightarrow +1 \\ \text{odd permutation} \rightarrow -1 \end{cases}$

any single permutation is like multiplying by (-1)

$\epsilon_{312} = ?$ $\quad \epsilon_{123}$

$$\epsilon_{321} = (-1)\,\epsilon_{123}$$

$$\epsilon_{312} = (-1)\,\epsilon_{321} = (-1)(-1)\,\epsilon_{123} = (-1)^2 \epsilon_{123} = 1$$

Uses of permutation symbol:

- Determinant of a matrix
- Inverse of a matrix
- Cross product
- Curl

Determinant of A:

$$\det A = \det\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = (-1)^{1+1} A_{11} \det\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}$$

$$+ (-1)^{1+2} A_{12} \det\begin{bmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{bmatrix}$$

$$+ (-1)^{1+3} A_{13} \det\begin{bmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}$$

$$\det A = A_{11}A_{22}A_{33} - A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{13}A_{22}A_{31}$$

Let's check $\epsilon_{ijk} A_{1i} A_{2j} A_{3k}$

write only the nonzero terms

$$\underbrace{\epsilon_{123}}_{1} A_{11} A_{22} A_{33} + \underbrace{\epsilon_{132}}_{-1} A_{11} A_{21} A_{32}$$

$$\underbrace{\epsilon_{213}}_{-1} A_{12} A_{21} A_{33} + \underbrace{\epsilon_{231}}_{1} A_{12} A_{23} A_{31}$$

$$+ \underbrace{\epsilon_{312}}_{1} A_{13} A_{21} A_{32} + \underbrace{\epsilon_{321}}_{-1} A_{13} A_{22} A_{31}$$

$$\boxed{\begin{aligned} \det A &= \epsilon_{ijk} A_{1i} A_{2j} A_{3k} \\ &= \epsilon_{ijk} A_{i1} A_{j2} A_{k3} \end{aligned}}$$