CM2023/08/30

Wednesday, August 30, 2023 9:46 AM

① $$\boxed{\begin{aligned}\det A &= \epsilon_{ijk} A_{1i} A_{2j} A_{3k} \\ &= \epsilon_{ijk} A_{i1} A_{j2} A_{k3}\end{aligned}}$$

HW1, you'll show

② $$\boxed{\begin{aligned}\epsilon_{mnp} \det A &= \epsilon_{ijk} A_{im} A_{jn} A_{kp} \\ &= \epsilon_{ijk} A_{mi} A_{nj} A_{pk}\end{aligned}}$$

m:3
n:3
p:3

27 equation's

$m=1 \quad n=2 \quad p=3 \qquad \epsilon_{123} \underline{\det A} = \epsilon_{ijk} A_{1i} A_{2j} A_{3k}$ eqn 1 $= \det B$

$m=1 \quad n=1 \quad p=3 \qquad 0 = \epsilon_{113} \det A = \epsilon_{ijk} A_{1i} A_{1j} A_{3k} = \epsilon_{ijk} B_{1i} B_{2j} B_{3k}$

equal

$$\text{RHS} = \det \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{11} & A_{12} & A_{13} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = 0 \text{ from } \star \text{ below}$$

$B_{1(\,)} = A_{1(\,)}$
$B_{2(\,)} = A_{1(\,)}$
$B_{3(\,)} = A_{3(\,)}$

$$\det \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}\begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} = -\det \begin{bmatrix} A_{21} & A_{22} & A_{23} \\ A_{11} & A_{12} & A_{13} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}\begin{matrix} r_2 \\ r_1 \\ r_3 \end{matrix}$$

every time we swap rows or columns of a matrix we get one (-1) multiplication

$$\det \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{11} & A_{12} & A_{13} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = -\det \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{11} & A_{12} & A_{13} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$2(\;''\;) = 0 \implies \det \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ \vdots & & \end{bmatrix} = 0 \quad \star$$

Properties of determinant:

1. If we swap two rows or two columns of matrix we get a (-1) factor for determinant

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{matrix} r_1(A) \\ r_2(A) \\ r_3(A) \end{matrix} = \begin{bmatrix} r_1(A) \\ \hline r_2(A) \\ \hline r_3(A) \end{bmatrix} = \left[C_1(A) \,\middle|\, C_2(A) \,\middle|\, C_3(A) \right]$$

$$\begin{matrix} C_1(A) & C_2(A) & C_3(A) \end{matrix}$$

e.g swap rows 1 & 3 :

$$\det \begin{bmatrix} r_3(A) \\ \hline r_2(A) \\ \hline r_1(A) \end{bmatrix} = (-1) \det \begin{bmatrix} r_1(A) \\ \hline r_2(A) \\ \hline r_3(A) \end{bmatrix}$$

that is

$$\det \begin{bmatrix} A_{31} & A_{32} & A_{33} \\ A_{21} & A_{22} & A_{23} \\ A_{11} & A_{12} & A_{13} \end{bmatrix} = (-1) \det \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

2. Based on 1 if two rows or columns of a matrix are equal det = 0

for A $\underline{A_{1i} = 0}$

$$\det \begin{bmatrix} 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = 0 \qquad \det A = \epsilon_{ijk} \underset{0}{A_{1i}} A_{2j} A_{3k} = 0$$

3. one row of a matrix is λ times the corresponding row of matrix A

$$C = \begin{bmatrix} \lambda A_{11} & \lambda A_{12} & \lambda A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \qquad \det C = \lambda \det A$$

$$\det C = \epsilon_{ijk} \underset{(\lambda A_{1i})}{C_{1i}} \underset{A_{2j}}{C_{2j}} \underset{A_{3k}}{C_{3k}} = \epsilon_{ijk} \lambda A_{1i} A_{2j} A_{3k}$$

$$= \lambda \underbrace{(\epsilon_{ijk} A_{1i} A_{2j} A_{3k})}_{\det A}$$

4. what about this

$$\underset{3\times3}{C} = \lambda A = \begin{bmatrix} \lambda A_{11} & & \\ & & \\ & & \lambda A_{33} \end{bmatrix} \qquad \det C = \lambda^3 \det A$$

$$A_{n\times n} \qquad \det(\lambda A) = \lambda^n \det A$$

5.

$$C = \begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} & A_{13}+B_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$\det C = \det \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & & \\ & & A_{33} \end{bmatrix} + \det \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ A_{22} & & \\ & & A_{33} \end{bmatrix}$$

$$\det C = \epsilon_{ijk}\, C_{1i}\, C_{2j}\, C_{3k} = \epsilon_{ijk}(A_{1i}+B_{1i})\, A_{2j}\, A_{3k}$$

$$(A_{1i}+B_{1j})\ A_{2j}\ A_{3k}$$

$$= \underbrace{\epsilon_{ijk}\, A_{1i}\, A_{2j}\, A_{3k}}_{\det A} + \epsilon_{ijk}\, B_{1i}\, A_{2j}\, A_{3k}$$

$$C = A + B = \begin{bmatrix} A_{11}+B_{11} & \cdots & A_{13}+B_{13} \\ A_{21}+B_{21} & & \\ & & A_{33}+B_{33} \end{bmatrix}$$

$$\det C \underset{\text{in general}}{\neq} \det A + \det B$$

6. $C = AB \qquad \underline{\det C = \det A \ \det B}$

$$\det C = \epsilon_{ijk}\, C_{1i}\, C_{2j}\, C_{3k} \qquad \begin{aligned} C_{1i} &= A_{1m} B_{mi} \\ C_{2j} &= A_{2n} B_{nj} \\ C_{3k} &= A_{3p} B_{pk} \end{aligned}$$

$C_{2j} = \epsilon_{2n} B_{nj}$

$C_{3k} = A_{3p} B_{pj}$

$= \det A \det B$

I think easier way is to use

$\epsilon_{mnp} \det C = \epsilon_{ijk} C_{mi} C_{nj} C_{pk}$

- indicial notation identity

ϵ_{ijk} ϵ_{mnp} multiply these

$(3)^6$ $\epsilon_{ijk} \epsilon_{mnp}$ let's contract k & p

$(3)^4$ $\boxed{\epsilon_{ijk} \epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}}$ (3) (*)

(3^2) terms

3x3 matrix

$\epsilon_{ijk} \epsilon_{mjk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$ another contraction

same as eq(3) but $n \to j$

$= \delta_{im}\underbrace{\delta_{jj}}_{3} - \delta_{ij}\delta_{jm} = 3\delta_{im} - \delta_{im} = 2\delta_{im}$

$\boxed{\epsilon_{ijk}\epsilon_{mjk} = 2\delta_{im}}$ (4)

Finally let's contract i & m

$\epsilon_{ijk}\epsilon_{ijk} = 2\delta_{ii} = 2\times3 = 6$ (5) $\boxed{\epsilon_{ijk}\epsilon_{ijk} = 6}$

$$\epsilon_{ijk}\,\epsilon_{ijk} = 2\,\delta_{ii} = 2\times 3 = 6 \quad (5) \quad \boxed{\epsilon_{ijk}\epsilon_{ijk} = 6}$$

$m \to i$ $\qquad \delta_{mm}$, $m \to i$

$$\epsilon_{mnp} \det A = \epsilon_{ijk} A_{im} A_{jn} A_{kp}$$

$\times\, \epsilon_{mnp}$

$$\underbrace{\epsilon_{mnp}\epsilon_{mnp}}_{6} \det A = \epsilon_{mnp}\epsilon_{ijk} A_{im} A_{jn} A_{kp}$$

All determinant formulas

(6)

$$\boxed{\begin{aligned} \det A &= \frac{1}{6}\epsilon_{mnp}\epsilon_{ijk} A_{im} A_{jn} A_{kp} \\ &= \epsilon_{ijk} A_{1i} A_{2j} A_{3k} = \epsilon_{ijk} A_{i1} A_{j2} A_{3k} \\ \& \ \epsilon_{mnp} \det A &= \epsilon_{ijk} A_{im} A_{jn} A_{kp} \end{aligned}}$$

suggest using one of these to prove $\det AB = (\det A)(\det B)$

Another indicial notation example:

$$\underset{\text{scalar value}}{Q} = \underbrace{[x_1 \;\; x_2 \;\; x_3]}_{x^t} \underbrace{\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}}_{\text{matrix } A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

→ vector x

e.g. energy densities

$$Q = x^t A x = x_i \left(\underline{(Ax)_i}\right) = \underline{x_i A_{ij} x_j}$$

dummy indices change them on

$$\frac{\partial Q}{\partial x_i} = ? \qquad \frac{\partial Q}{\partial x_i} = \frac{\partial (x_i A_{ij} x_j)}{\partial x_i}$$

1 mistake here

3 indices i

$$= A_{ij} x_j$$

& another one here

$=$

$$\frac{\partial Q}{\partial x_i} = \frac{\partial\, x_m A_{mj} x_j}{\partial x_i} =$$

$$\frac{\partial x_m}{\partial x_i} A_{mj} x_j + x_m \cancel{\frac{\partial A_{mj}}{\partial x_i}}^{\,0} x_j + x_m A_{mj} \frac{\partial x_j}{\partial x_i}$$

$$\delta_{mi} A_{mj} x_j \qquad + x_m A_{mj} \delta_{ji}$$

$$A_{ij} x_j \qquad + \underbrace{x_m A_{mi}}_{\text{dummy}}$$

$$= A_{ij} x_j + x_j A_{ji}$$

$$= (A_{ij} + A_{ji})\, x_j$$

$$= (A_{ij} + (A^t)_{ij})\, x_j$$

$$= 2\left(\frac{A + A^t}{2}\right)_{ij} x_j$$

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$A^t = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$(A^t)_{ij} = A_{ji}$$

summary

$$Q = x \cdot A x \qquad \frac{\partial Q}{\partial x_i} = 2\left(\frac{A + A^t}{2}\right)_{ij} x_j$$

or

$$\frac{\partial Q}{\partial x} = 2\underbrace{\left(\frac{A + A^t}{2}\right)}_{\text{sym } A} x = 2\,(\text{sym } A)\, x$$

scalar T 0th order tensor

kinematic / $\vec{u}$ disp vectors

$\vec{}$ velocity

Vector spaces

Motivation: Why do we care about vector spaces?

Vector spaces

Motivation: Why do we care about vector spaces?

kinematic $\left[\begin{array}{ll} \vec{u} & \text{disp} \\ \vec{v} & \text{velocity} \end{array}\right.$ vectors, 1st order tensor

balance laws $\left[\begin{array}{ll} \epsilon & \text{strain} \\ \sigma & \text{stress} \end{array}\right.$ 2nd order tensor

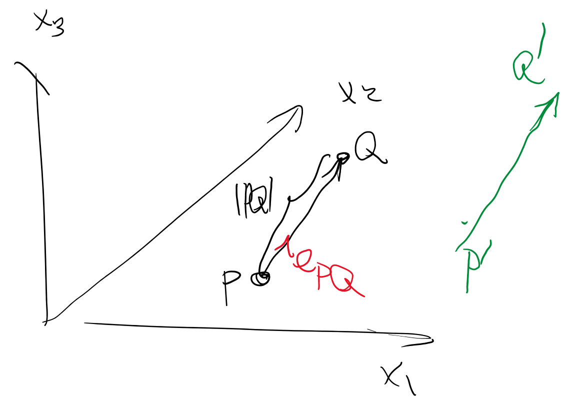
Const eqn $\left[\ \sigma = C\,\varepsilon \right.$ → C: 4th order elasticity tensor

$$\sigma_{ij} = C_{ijkl}\,\epsilon_{kl}$$

All of these are based on the concept of vector spaces

A vector $\overrightarrow{PQ}$ is identified by 3 things:

1. Length of PQ: $|PQ|$
2. Direction of PQ: e_{PQ}



3. Sometimes we also care about the BASE of the vector, which is point P here:
   a. If we differentiate vectors based on their base points, we call them bound vectors
   b. If not, they are free vectors

$PQ = P'Q'$ even though $P = P'$