CM2023/09/06

Wednesday, September 6, 2023 9:32 AM

From the course webpage:

Resources

- **Equation sheet**: Credit to my colleague Dr. Scott Miller for this material. The formulation sheet will be updated throughout the course.

$$\epsilon_{ijk}\epsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp} = \begin{vmatrix} \delta_{jp} & \delta_{jq} \\ \delta_{kp} & \delta_{kq} \end{vmatrix}$$

TAM551.pdf

Theorem 11

1. $\varepsilon_{ijk}\varepsilon_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix}$,
2. $\varepsilon_{ijk}\varepsilon_{iqr} = \delta_{jq}\delta_{kr} - \delta_{jr}\delta_{kq}$, (this result is worth remembering)
3. $\varepsilon_{ijk}\varepsilon_{ijr} = 2\delta_{kr}$,
4. $\varepsilon_{ijk}\varepsilon_{ijk} = 6$.

Proof. Set $a_{mn} = \delta_{mn}$ in Theorem 10. ■

More mechanical insight from

Lecture Notes on The Mechanics of Elastic Solids

Volume 1: A Brief Review of Some Mathematical Preliminaries

There are a number of Worked Examples at the end of each chapter which are an essential part of the notes. Many of these examples either provide, more details, or a proof, of a result that had been quoted previously in the text; or it illustrates a general concept; or it establishes a result that will be used subsequently (possibly in a later volume).

Continue from last time

VECTOR SPACES

real number,

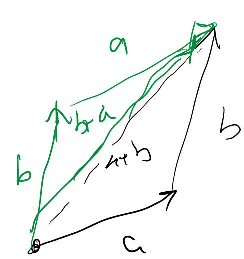
Properties of vector spaces $a, b \in V$, $\lambda, \mu \in \mathbb{R}$

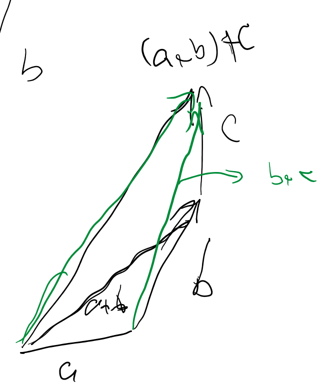
ADDITION: $(V, +, \cdot)$

A1) $a + b = b + a$ commutative

A2) $a + (b + c) = (a + b) + c$ associative

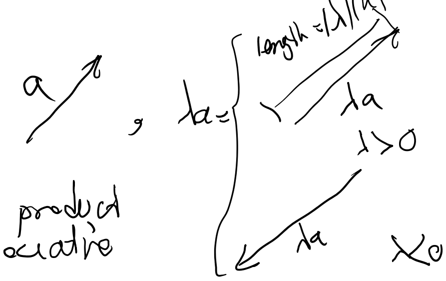
A3) $a + 0 = 0 + a = a$ zero vector





$$(A3)\quad a + 0 = 0 + a = a \qquad \text{zero vector}$$

SCALAR PRODUCT PROPERTY $\quad \lambda \in \mathbb{R},\ a,\ \lambda a =$



side note
Group has properties 1 to 3 plus (negative/inverse) property
(4) $a + (-a) = (-a) + a = 0$
we can prove (4) for vector spaces

Basically vector spaces are Groups for (+) property

P1) $(\lambda \mu) a = \lambda(\mu a)$ scalar product associative

P2) $(\lambda + \mu) a = \lambda a + \mu a$ distributive w.r.t. scalar addition

P3) $\lambda(a+b) = \lambda a + \lambda b$ distributive w.r.t. vector addition

P4) $1 \cdot a = a$

Prove $a + (-a) = 0 \qquad -a := (-1)a$ Definition

$$a + -a = a + (-1)a$$
$$= 1a + (-1)a \quad (P4)$$
$$= (1 + -1)\, a \quad (P2)$$
$$= 0 \cdot a$$
$$= 0$$

why?

$$0 \cdot a + a = 0 \cdot a + 1 \cdot a \ (P4) = (0+1)\,a \ (P2) = 1a = a \ (P4)$$

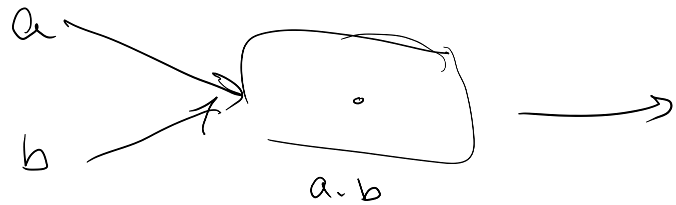
from (A3) $\quad 0 + a = a$

$$0 \cdot a = 0$$

Inner product between two vectors
Also called scalar product of vectors

Also called scalar product of vectors

vectors a, b



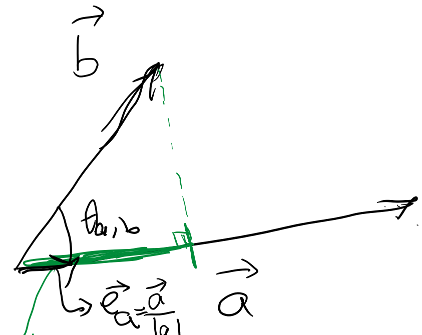
scalar value

$a \cdot b$ is a scalar value

eg.

$P_a(b)$ = length of the green segment

$= |b| \cos\theta_{a,b}$



$|e_a| = \frac{|a|}{|a|} = 1$

projection of vector b on a

inner product is defined as

$$\boxed{a \cdot b = |a||b| \cos\theta_{a,b} = |a| P_a(b) = |b| P_b(a)} \quad (1)$$

For projection if the orientation of green line is needed as well we do the following

$$\vec{P_a}(b) = P_a(b)\,(\text{orientation of } a) = P_a(b) \frac{\vec{a}}{|a|} = \frac{|b| \cos\theta_{a,b}}{|a|} \cdot \frac{|a|}{|a|} \vec{a}$$

$$= \frac{|a||b| \cos\theta_{a,b}}{|a||a|} \vec{a} = \frac{a \cdot b}{|a||a|} = \frac{a \cdot b}{a \cdot a} \vec{a}$$

what is $|a||a|$?

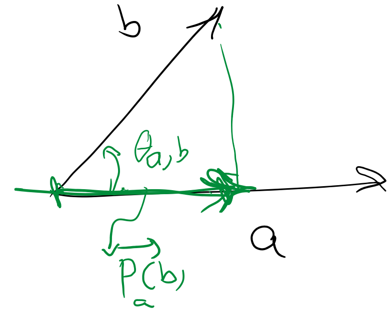
$$\vec{a} \cdot \vec{a} = |a||a| \underbrace{\cos\theta_{a,a}}_{\substack{\theta \\ 1}} = |a||a| = |a|^2$$

Summary of relation between inner product & projection

(2)

$$\overrightarrow{P_a(b)} = \underbrace{\left(\frac{a \cdot b}{a \cdot a}\right)}_{\text{scalar number}} \vec{a}$$

vector

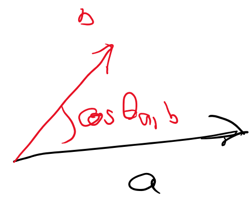


$$P_a(b) = \left| \overrightarrow{P_a(b)} \right| = |b| \cos\theta_{a,b}$$

$$= \frac{a \cdot b}{|a|}$$

Some interesting facts about .

$a \cdot b = |a||b| \cos\theta_{a,b}$

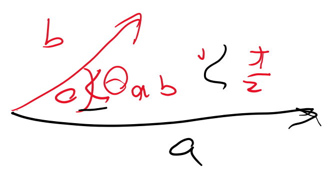


1) $\theta_{a,b} = 0$ Maximum (a,b)

$a \cdot b = |a||b|$

2) $-\frac{\pi}{2} < \theta_{a,b} < \frac{\pi}{2}$

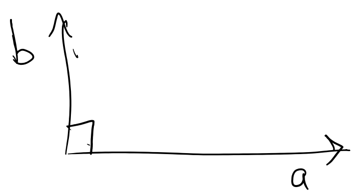
$a \cdot b > 0$



$a \cdot b > 0$

3) $\theta_{a,b} = \frac{\pi}{2}$

$a \cdot b = 0$

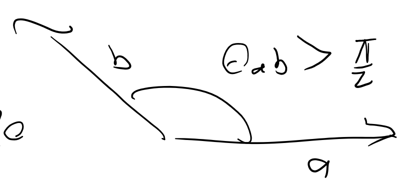


(3) $\boxed{a \cdot b = 0 \iff a \perp b}$

A good way to check if two vectors are normal

4) $\theta_{a,b} > \frac{\pi}{2}$

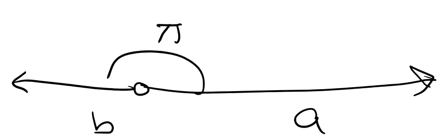
$a \cdot b < 0$ $\cos\theta_{a,b} < 0$



5) $\theta_{a,b} = \pi$

$a \cdot b = |a||b| \cos\pi =$

$-|a||b|$ Minimum (a,b)



Properties of inner product $a, b, c \in V$ (vectors), $\lambda \in R$

(4)

I.1 $a \circ (\lambda b) = (\lambda a) \cdot b = \lambda (a \cdot b)$ scalar product homogeneity

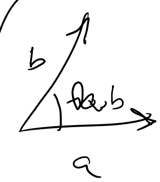
I.2 $a \cdot (b + c) = a \cdot b + a \cdot c$ Distributive w.r.t vector addition

I.3 $a \cdot b = b \cdot a$ commutative property for .

I.4 $a \cdot a \geq 0$ &

$a \cdot a = 0 \iff a = 0$

positive definite property



$a \cdot b = |a||b| \cos\theta_{a,b}$
$= |b||a| \cos(-\theta_{b,a})$
$= |b||a| \cos\theta_{b,a}$
$= b \cdot a$

a

$a \cdot a = |a|^2$

motivation for real vector

$(I4) \xrightarrow{\text{def}} |a| = \sqrt{a \cdot a}$ in general

- A vector space (V, +, scalar product that satisfies properties A1-3, P1-4) that also has inner product (properties I1 to I4) is called an inner **product vector space**.
- Not all vector spaces have an inner product.
- Notations for inner product:

$$a \circ b \qquad (a, b) \qquad \langle a, b \rangle$$

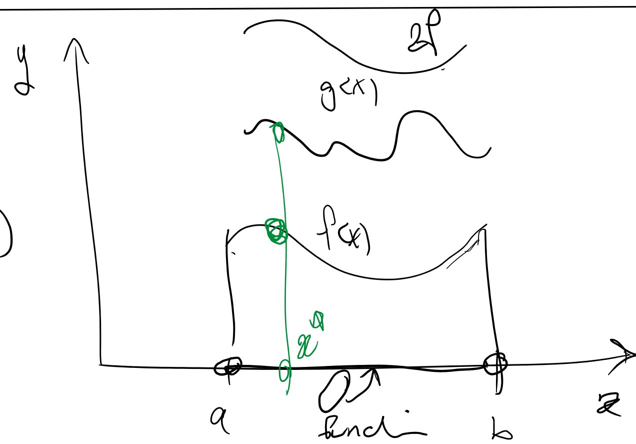
A nontrivial example of vector spaces:

Functions defined on (a, b)

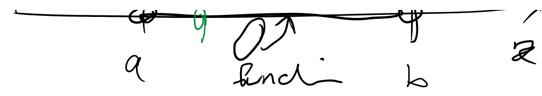
we need to define vector addition $(f+g)$

$f + g = ?$

$\forall x \in (a, b) \quad (f+g)(x) \overset{\text{defined as}}{=} f(x) + g(x)$



$\forall x \in (a,b)$ $(f+g)(x) := f(x) + g(x)$



& scalar product

$\lambda f = ?$

$0(x) = 0$

zero function

number 0

$\forall x \in (a,b)$ $(\lambda f)(x) := \lambda f(x)$

Now we need to prove that this is a vector space (A1 - A3, P1 to P4)

example A1 $a+b = b+a$

$f+g = g+f$?

$\underline{\forall x \in (a,b)}$ $(f+g)(x) = f(x) + g(x)$ def of $f+g$

$= g(x) + f(x)$

$= (g+f)(x)$ def of $g+f$

so $f+g = g+f$

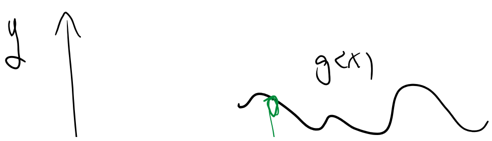
other properties are similarly proved by $\forall x \in (a,b)$ ...

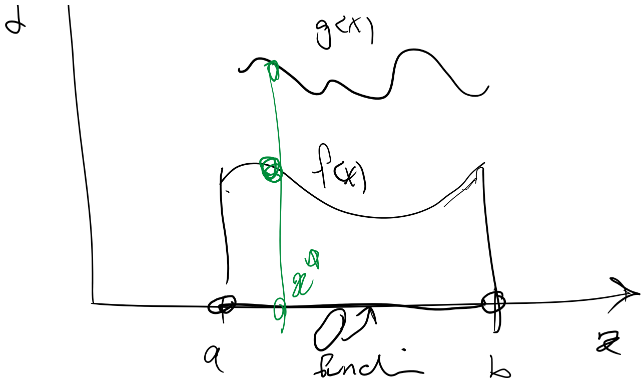
use real # properties

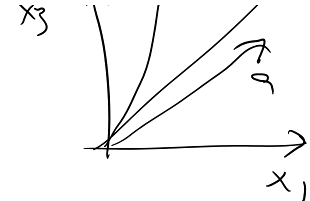
Do functions have an inner product?

Note I'll show for vectors $a \cdot b = a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$ (in 3D)

For functions:







$$\boxed{\langle f,g\rangle = \int_a^b f(x)\,g(x)\,dx}$$

(5) for functions

Is this an inner product?

need to check I1 to I4 are satisfied

(4)
- I.1 $\quad a\cdot(\lambda b) = (\lambda a)\cdot b = \lambda(a\cdot b)$
- I.2 $\quad a\cdot(b+c) = a\cdot b + a\cdot c$
- I.3 $\quad a\cdot b = b\cdot a$
- I.4 $\quad a\cdot a \geq 0$ & $a\cdot a = 0 \iff a = 0$

Yes :)

(6) for functions that are "square integrable"

$$f\cdot f = \int_a^b f(x)\cdot f(x)\,dx = \int_a^b f^2(x)\,dx < \infty$$

we have an inner product vector space

$$V = \{\text{functions defined on } (a,b)\}$$

$$W = \{\ \text{''}\quad \text{''}\quad \text{''},\ \underbrace{\int_a^b f^2\,dx}_{\text{finite integral}} < \infty\} = L_2(a,b)$$

W has inner product, V doesn't

$$\underbrace{W \subseteq V}_{\text{subset of } V}$$

why I cannot define $\langle \, , \, \rangle$ for $\mathcal{U}$

Example $\quad f(x) = \dfrac{1}{\sqrt{x}}$

$$\langle f, f \rangle = \int_0^1 \frac{1}{\sqrt{x}} \frac{1}{\sqrt{x}} = \int_0^1 \frac{1}{x} \, dx = \ln x \Big|_0^1 = \infty$$