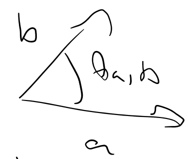
CM2023/09/08

Friday, September 8, 2023 9:35 AM

Recall for real vectors



$$a \cdot b = |a||b| \cos\theta_{a,b}$$

$$\cos\theta_{a,b} = \frac{a \cdot b}{|a||b|} \qquad |\cos\theta_{a,b}| = \frac{|a \cdot b|}{|a||b|} \leq 1$$

$\rightarrow$ (1) $\underbrace{|a \cdot b| \leq |a||b|}_{\text{Cauchy-Schwarz inequality}}$ for real vectors

In fact, for a general definition of an inner product, we can prove (1)

$\cdot$ is inner product $\qquad |a \cdot b| \leq |a||b|$

I1 $\quad f \cdot g = g \cdot f$

I2 $\quad (f+g) \cdot h = f \cdot h + g \cdot h \rightarrow h \cdot (f+g) = h \cdot f + h \cdot g$

I3 $\quad (\lambda f) \cdot g = \lambda\, f \cdot g$

I4 $\quad |f|^2 = f \cdot f \geq 0 \qquad f \cdot f = 0$ iff $f = 0$

Proof $\qquad |f \cdot g| \leq |f||g|$ for any two vectors f & g

$$(f + \alpha g) \cdot (f + \alpha g) \geq 0 \qquad \text{I4}$$

$\alpha \in R$

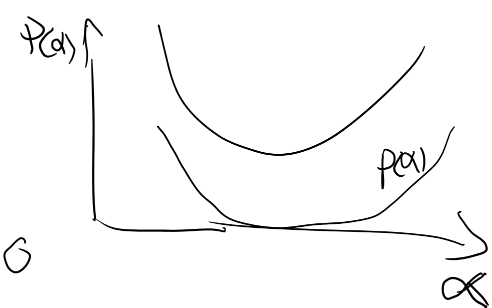
$$(f + \alpha g) \cdot f + (f + \alpha g) \cdot (\alpha g) \geq 0 \quad \text{I2}$$

$$f \cdot f + (\alpha g) \cdot f + f \cdot (\alpha g) + (\alpha g) \cdot (\alpha g) \geq 0 \quad \text{I2}$$

$$f \cdot f + \alpha (g \cdot f) + \alpha (f \cdot g) + \alpha (g \cdot (\alpha g)) \geq 0 \quad \text{I3}$$

I1

$$f \cdot f + \alpha (f \cdot g) + \alpha (f \cdot g) + \underbrace{\alpha^2}_{\text{I3}} (g \cdot g) \geq 0$$



$$f \cdot f + 2\alpha\, f \cdot g + \alpha^2\, g \cdot g \geq 0 \qquad \forall \alpha \in \mathbb{R}$$

$$(2) \quad \boxed{\underbrace{|f|^2}_{c_0} + 2\alpha\, \underbrace{f \cdot g}_{c_1\,(2 f \cdot g)} + \alpha^2 \underbrace{|g|^2}_{c_2} \geq 0}$$

$$P(\alpha) = c_0 + c_1 \alpha + c_2 \alpha^2 \geq 0$$

$$\Delta = c_1^2 - 4 c_0 c_2 \leq 0$$

$$(2\, f \cdot g)^2 - 4|f|^2 |g|^2 \leq 0$$

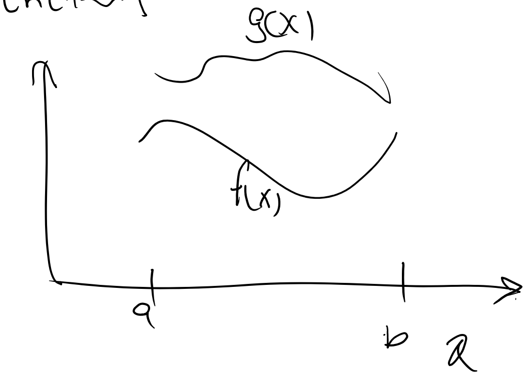
$$0 \leq 4 (f \cdot g)^2 \leq 4|f|^2 |g|^2 \qquad \text{take square root}$$

$0 \leq 4(f\cdot g)^2 \leq 4|f|^2|g|^2$ take square root

$$\boxed{|f\cdot g| \leq |f||g|} \quad (1)$$

Cauchy-Schwarz (CS)

Using Cauchy Schwarz (CS) inequality we can now define angle between "general vectors" such as functions



$$\boxed{\cos\theta_{fg} = \frac{f\cdot g}{|f||g|}} \quad (3)$$

from (CS) we know $\left|\frac{f\cdot g}{|f||g|}\right| \leq 1$

so cosine definition makes sense

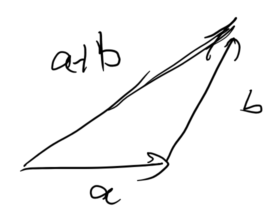
with inner product we can define

(1) magnitudes $|f| = \sqrt{f\cdot f}$

(2) Angles $\cos\theta_{fg} = \frac{f\cdot g}{|f||g|}$

HW: Triangular inequality

$$\boxed{(4) \quad |a+b| \leq |a| + |b|}$$



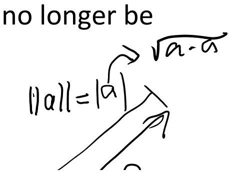
Hint square eq (4) & use CS inequality after that

$|a+b|^2 = (a+b)\cdot(a+b) = a\cdot a + \text{—} \;\; \text{—}$

Norm:

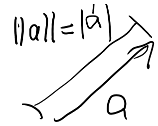
Norm is a weaker condition, by which we can only define magnitudes (called norm) but angle can no longer be defined

$a \longrightarrow$ norm $\|a\|$ norm of a



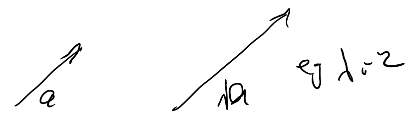
$a \longrightarrow$ norm $\|a\|$ norm of a

this is a scalar number



for real vectors

$\boxed{\begin{array}{ll} N1) & \|\lambda a\| = |\lambda|\,\|a\| \\ N2) & \|a\| \geq 0 \quad \& \quad \|a\| = 0 \text{ iff } a = 0 \\ N3) & \|a+b\| \leq \|a\| + \|b\| \quad \text{triangular inequality} \end{array}}$ (5)



Is an innerproduct induced magnitude actually a norm?

- is defined is $\|f\| = \sqrt{f \cdot f}$ * this a norm?

N1) $|\lambda a| = |\lambda|\,|a|$?

N2) $|a| \geq 0$ & $|a| = 0$ iff $a = 0$

N3) $|a+b| \leq |a| + |b|$

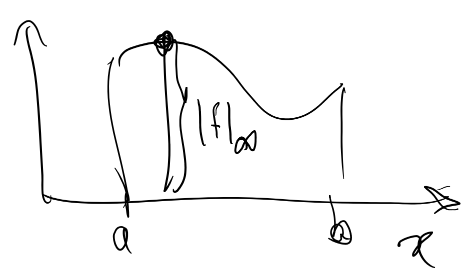
Norm only corresponds to the definition of a magnitude of a vector (① for (·) above) but cannot capture property ② of inner product (angle)

Inner product $(\cdot)$ $\longrightarrow$ $|a| = \sqrt{a \cdot a}$ norm

Cannot define an inner product $\cancel{\longleftarrow}$ $\|a\|$

A few examples of norms:

$$\|f\|_p := \sqrt[p]{\int_a^b |f(x)|^p \, dx}$$



e.g.
$$\begin{cases} \|f\|_1 = \int_a^b |f(x)|\,dx & L1 \text{ norm} \\ \|f_2\| = \sqrt{\int |f(x)|^2\,dx} & L2 \text{ ''} \\ \|f\|_\infty = \lim_{p\to\infty} \|f\|_p = \max_{x\in(a,b)} f(x) & L\infty \text{ norm} \end{cases}$$

Came from an inner product
$f \cdot g = \int_a^b f \cdot g\,dx$
$\rightarrow |f| = \sqrt{f \cdot f} = \sqrt{\int |f|^2 dx}$

Best option

$\frac{1}{x^{1/4}} \in L_2(0,5)$
not bounded $\notin L_\infty(0,5)$

FYI:

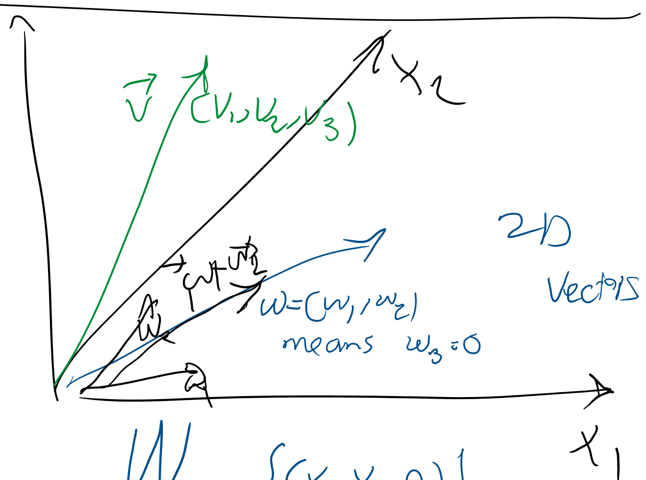
Subspace

Vectors in 3D from a vector space
they satisfy all these 7 properties

$$\begin{cases} A1) & a+b = b+a & \text{commutative} \\ A2) & a+(b+c) = (a+b)+c & \text{associative} \\ A3) & a+0 = 0+a = a & \text{zero vector} \end{cases}$$

SCALAR PRODUCT PROPERTY $\lambda \in R$, a, λa length $= |\lambda| \|a\|$, λa $\lambda > 0$, λa

$$\begin{cases} P1) & (\lambda\mu)a = \lambda(\mu a) & \text{scalar product associative} \\ P2) & (\lambda+\mu)a = \lambda a + \mu a & \text{distributive w.r.t. scalar addition} \\ P3) & \lambda(a+b) = \lambda a + \lambda b & \text{distributive w.r.t. vector addition} \\ P4) & 1 \cdot a = a & \end{cases}$$



$W = \{(x_1, x_2, 0)\}$ 2D plane

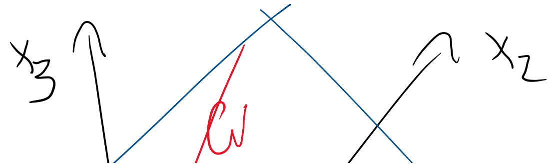
$V = \{(x_1, x_2, x_3)\}$ 3D space

- V is a vector space
- $W \subseteq V$ (W subset of V)
- $\forall \omega_1, \omega_2 \in W$ & $\lambda \in R$: $\omega_1 + \lambda\omega_2 \in W$

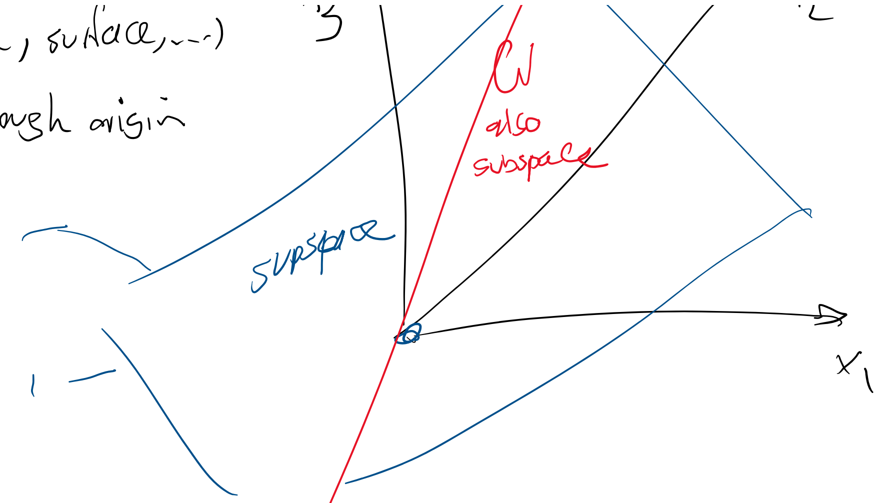
W is closed with respect of addition $\omega_1 + \omega_2 \in W$
scalar product $\lambda \omega_1 \in W$

then W is a vector space

any flat "plane" (line, surface, ...)

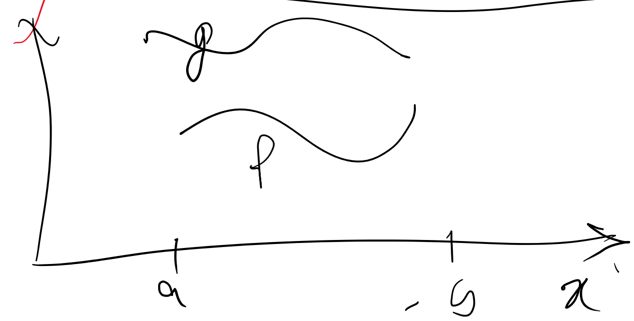


any flat "plane" (line, surface, ...)
subset of V that goes through origin
is a subspace.



Example

$V = \{$ functions defined over $(a,b)\}$
was a vector space



$$W = \left\{ f \in V \;\middle|\; \underbrace{\int_a^b |f|^2 dx}_{\|f\|_2} < \infty \right\}$$

functions that have L_2 norm

is W a vector space?

- V is a vector space
- $W \subseteq V$
- $w_1, w_2 \in W \rightarrow w_1 + w_2 \in W$
- $w_1 \in W \quad \lambda w_1 \in W$
  $\lambda \in W$

$w_1 \in W \quad \|w_1\|_2 < \infty$
$w_2 \in W \quad \|w_2\|_2 < \infty$
want to show
$w_1 + w_2 \in W$
that is $\|w_1 + w_2\|_2 < \infty$
this is true because of triangle inequality
$\|w_1 + w_2\|_2 < \|w_1\|_2 + \|w_2\|_2$

$w_1 \in W \Rightarrow \|w_1\|_2 < \infty$

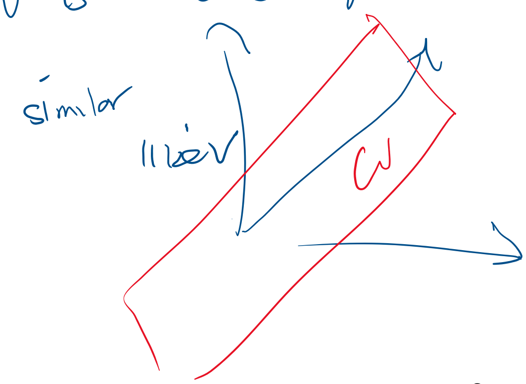
want to show

$\lambda w_1 \in W$ that is $\|\lambda w_1\|_2 < \infty$

But $\|\lambda w_2\|_2 = |\lambda| \|w_1\|_2 < \infty$

But $\|\lambda w_2\|_2 = |\lambda| \, \|w_2\|_2 < \infty$

W is a subspace of V that interestingly lp has a norm :)



In fact L_2 space is even stronger & is an inner product space

$$f \cdot g = \int_a^b f(x) \cdot g(x) \, dx$$

$$f \in W \Rightarrow |f| = \sqrt{\int_a^b f \cdot f \, dx} < \infty$$

$$g \in W \Rightarrow |g| = \sqrt{\int_a^b g \cdot g \, dx} < \infty$$

? $|f \cdot g| \le |f| \, |g| < \infty$

$f \cdot g = \int_a^b f(x) g(x) \, dx$ exists & is finite

Coordinate systems and coordinate transformation:

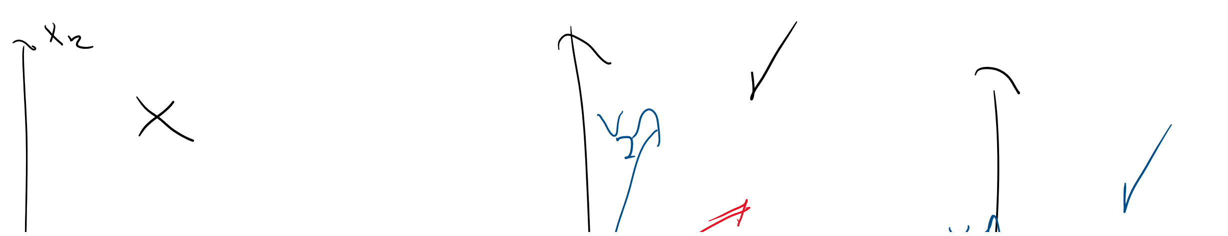
Linear independence:

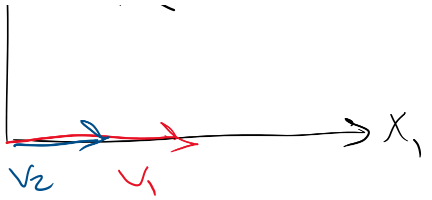
vectors $v_1, \dots, v_n$ are called linearly independent if

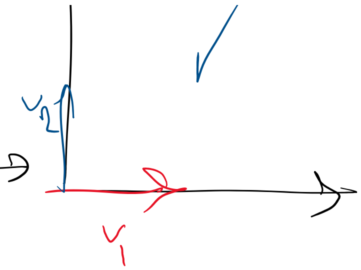
- for all $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ if

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0 \Longrightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$$

2D







$= .5 v_1$

such that

there $\exists\ \alpha_1, \alpha_2$

$\alpha_1 v_1 + \alpha_2 v_2 = 0$ & α_1, α_2 are not all zero

$.5\ v_1 + (-1) v_2 = 0$

Basis vector $e = \{e_1, \ldots, e_n\}$ for a vector space V is a basis if

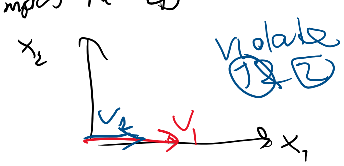
1. $e_1, \ldots, e_n$ are linearly independent → Unique coordinate values
2. Any vector $v \in V$ can be expressed as a linear combination of $\{e_1, \ldots, e_n\}$ → every vector has a coordinate

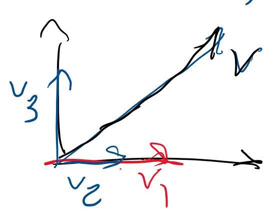
$V = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

coordinate of v w.r.t. e is $\Longleftrightarrow$

$(\alpha_1, \ldots, \alpha_n)$

Examples in 2D

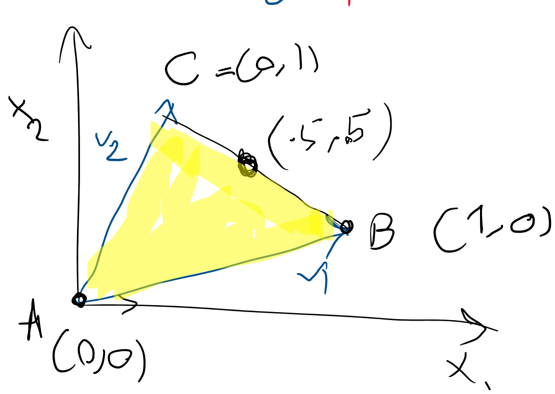




$v = 1.2 v_1 + 1.5 v_3$ ✓

$v = 2.4\ v_2 + 1.5 v_3$

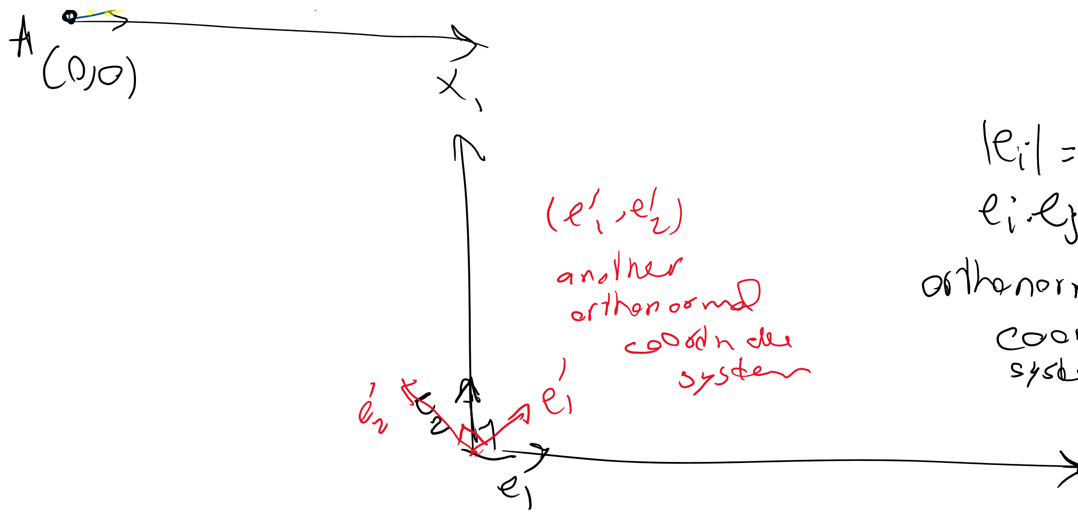
natural coordinate



$B = 1 V_1 + 0 V_2$

coordinate

for triangle



(e'_1, e'_2)
another
orthonormal
coordinate
system

$|e_i| = 1$
$e_i \cdot e_j = \delta_{ij}$
orthonormal
coordinate
system