CM2023/09/11

Monday, September 11, 2023 9:47 AM

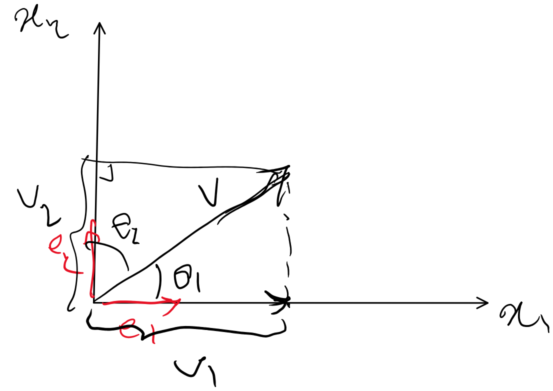
Orthonormal coordinate systems:

$e_i \cdot e_j = \delta_{ij}$

$|e_1| = 1$

$|e_2| = 1$

$e_1 \cdot e_2 = 0 \quad (e_1 \perp e_2)$



$V = V_i e_i$

coordinates of V

$V \cdot e_j = (V_i e_i) \cdot e_j = V_i (e_i \cdot e_j)$

$(a+b) \cdot c = a \cdot b + a \cdot c$

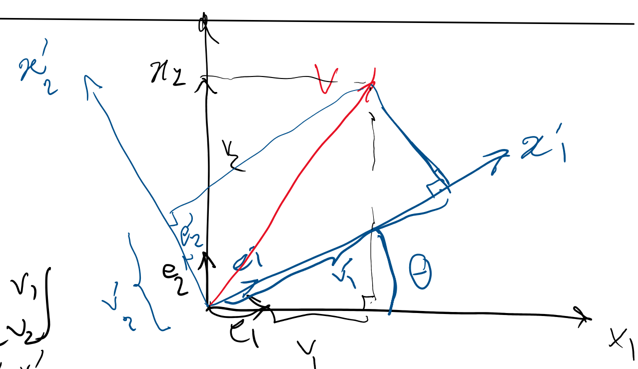
$= V_i \delta_{ij}$

$= V_j$

(1)

$$\boxed{\begin{aligned} V_j &= V \cdot e_j \\ V &= V_j e_j = P_{e_j} V \\ &= |V| \cos\theta_j \end{aligned}}$$

Coordinate transformation:



$V = V_1 e_1 + V_2 e_2 = V_i e_i$

$= V'_1 e'_1 + V'_2 e'_2 = V'_i e'_i$

V_i relationship V'_i $\begin{bmatrix} V'_1 \\ V'_2 \end{bmatrix} \overset{?}{=} \text{matrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$

or $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \overset{?}{=} \text{matrix} \begin{bmatrix} V'_1 \\ V'_2 \end{bmatrix}$

- both coordinate systems $\{e\}$ & $\{e'\}$ are orthonormal ($e_i \cdot e_j = \delta_{ij}$, $e'_i \cdot e'_j = \delta_{ij}$)

- we also know

$$\boxed{e'_i = Q_{ij} e_j}$$

Q: coordinate transformation matrix

TAM551 notes, λ is used instead of Q

what's the meaning of Q_{ij}

$e'_i = Q_{ij} e_j \quad \cdot e_k$ dot product with e_k on both sides

$e'_i \cdot e_k = (Q_{ij} e_j) \cdot e_k = Q_{ij} (e_j \cdot e_k) = Q_{ij} \delta_{jk} = Q_{ik}$

δ_{jk}

$(a+b) \cdot c = a \cdot b + a \cdot c$

$$e'_i \cdot e_k = (Q_{ij} e_j) \cdot e_k = Q_{ij}\, e_j \cdot e_k = Q_{ij}\, \delta_{jk} = Q_{ik}$$

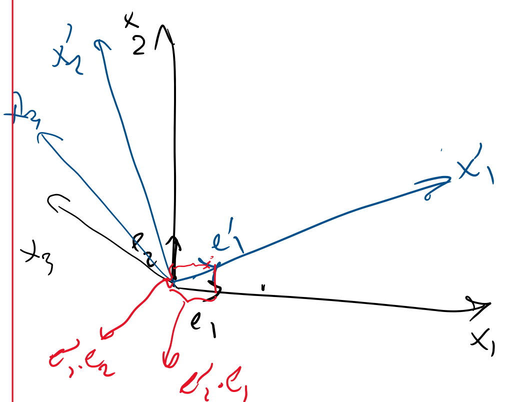
$(a+b).c = a.b + a.c$

$$Q_{ik} = e'_i \cdot e_k \qquad (k \to j)$$

$$e'_i = Q_{ij}\, e_j \iff Q_{ij} = e'_i \cdot e_j = \text{component } \underline{j} \text{ of vector } e'_i \text{ in coordinate } (e_1, e_2, \dots)$$

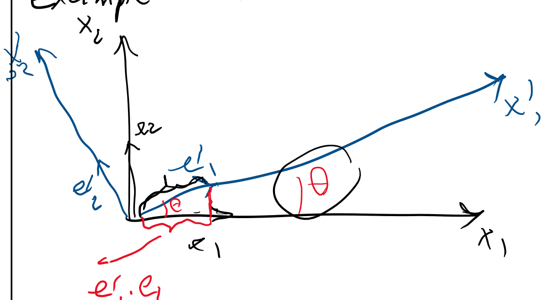
Recall eqn (1) $v_j = v \cdot e_j$

here $v = e'_i$



$$\underbrace{Q_{3\times3}}_{3D} = \begin{bmatrix} e'_1 \cdot e_1 & e'_1 \cdot e_2 & e'_1 \cdot e_3 \\ e'_2 \cdot e_1 & e'_2 \cdot e_2 & e'_2 \cdot e_3 \\ e'_3 \cdot e_1 & e'_3 \cdot e_2 & e'_3 \cdot e_3 \end{bmatrix} = \begin{bmatrix} \text{components of } e'_1 \text{ in } e \text{ system} \\ \text{" " } e'_2 \text{ in } e \text{ system} \\ \text{" " } e'_3 \text{ in } e \text{ system} \end{bmatrix} \quad (2)$$

Example: 2D



$$Q = \begin{bmatrix} e'_1 \text{ in } e \text{ system} \\ e'_2 \text{ in } e \text{ system} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad (3)$$

for 2D

$$= \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

$c := \cos\theta$

$s := \sin\theta$

Before showing how Q is used for coordinate transformation, we discuss an interesting property of Q

$$e'_i = Q_{ij}\, e_j \iff Q_{ij} = e'_i \cdot e_j$$

$$e_j = R_{ji}\, e'_i \longrightarrow \underline{R_{ji}} = e_j \cdot e'_i = e'_i \cdot e_j = \underline{Q_{ij}}$$

$$\boxed{\begin{aligned} e'_i &= Q_{ij}\, e_j \\ e_j &= R_{ji}\, e'_i \end{aligned}} \quad (4)$$

$$\begin{bmatrix} e'_1 \\ e'_2 \\ e'_3 \end{bmatrix} = Q \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \implies$$

$$\boxed{\begin{aligned} e_j &= R_{ji}\, e'_i \\ &= Q_{ij}\, e'_i \end{aligned}}$$

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \underline{Q^{-1}} \begin{bmatrix} e'_1 \\ e'_2 \\ e'_3 \end{bmatrix}$$

$$e_j = Q_{ij}\, e'_i = (Q^t)_{ji}\, e'_i \qquad \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = Q^t \begin{bmatrix} e'_1 \\ e'_2 \\ e'_3 \end{bmatrix}$$

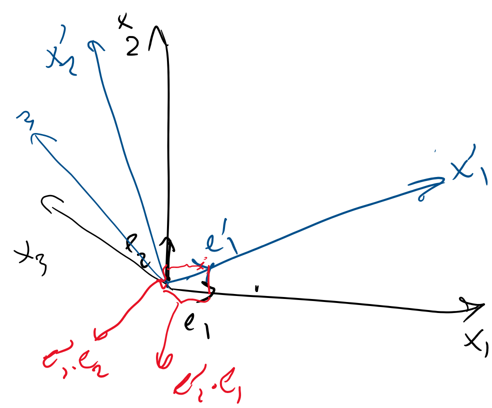
$$Q^{-1} = Q^t \quad \text{or} \quad Q Q^{-1} = Q Q^t = \underset{\text{Identity}}{I}$$

Q is an orthogonal matrix

(5) $$\boxed{Q Q^t = Q^t Q = I}$$

Another way to observe this

$$Q = \begin{pmatrix} e'_1 \\ \hline e'_2 \\ \hline e'_3 \end{pmatrix} \text{ expressed in } (e_1, e_2, e_3) \text{ coordinate system}$$



$$Q Q^t = \begin{bmatrix} e'_1 \\ \hline e'_2 \\ \hline e'_3 \end{bmatrix} \begin{bmatrix} e'_1 & e'_2 & e'_3 \end{bmatrix}$$

row 2 col 3

$$Q = \begin{bmatrix} \text{Components of } e'_1 \text{ in } e \text{ system} \\ \hline \text{" " } e'_2 \text{ in } e \text{ system} \\ \hline \text{" " } e'_3 \text{ in } e \text{ system} \end{bmatrix}$$

$$(Q Q^t)_{23} = e'_2 \cdot e'_3 = 0$$

$$(Q Q^t)_{22} = e'_2 \cdot e'_2 = 1$$

$$Q Q^t = I$$

because we're dealing with the orthonormal coordinate transformation

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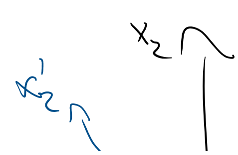
What's the use of Q matrix?

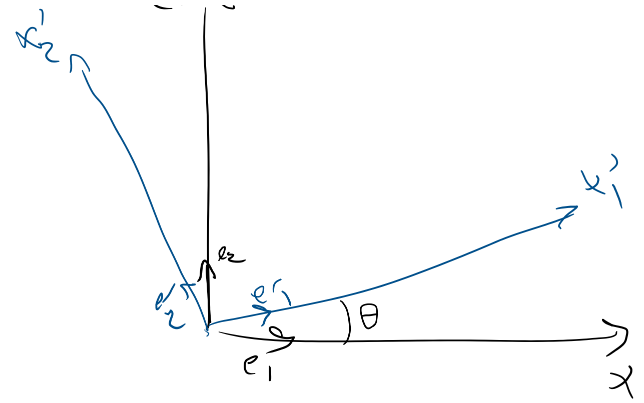
$V = V_i e_i$ Goal



$V = V_j e_j$

Goal
$v = v'_i e'_i$

How are v'_i & v_j related?



$$\left. \begin{array}{l} v = v_j e_j \\ e_j = Q_{ij} e'_i \end{array} \right\} \quad v = v_j (Q_{ij} e'_i)$$

$$= (v_j Q_{ij}) e'_i$$

$$= (Q_{ij} v_j) e'_i = v'_i e'_i$$

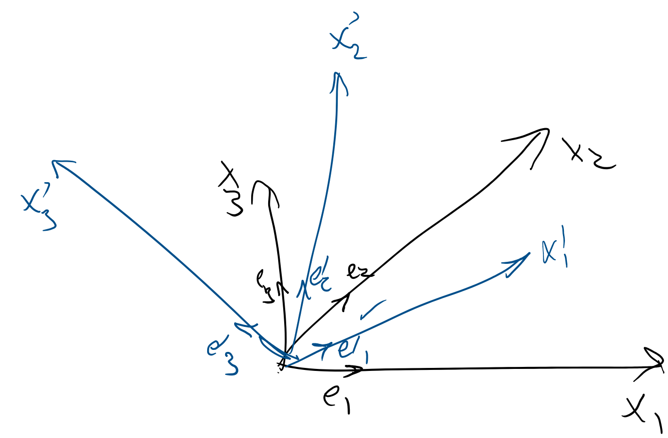
Recall $e'_i = Q_{ij} e_j$
&
$e_j = Q_{ij} e'_i$

(6) $\boxed{v'_i = Q_{ij} v_j}$

Summary

$$Q = \left[\begin{array}{c} \text{Comp of } e'_1 \\ \hline \text{"} \quad e'_2 \\ \hline \text{"} \quad e'_3 \end{array} \right] \text{ in } e \text{ sys}$$

$$= \left([\, e_1 \,|\, e_2 \,|\, e_3 \,] \right) \text{ in } e' \text{ system}$$



scalar $\quad T' = T$

vector $\quad V'_i = Q_{ij} V_j$

$[v'] = Q[v]$
components of v' $\quad$ components of $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

$V_j = Q_{ij} V'_i$

$[v] = Q^t [v']$
$= Q^{-1} [v']$
$(Q^t = Q^{-1})$

2nd order tensors

$T'_{ij} = Q\,Q\,T_{mn}$:

$T'_{ij} = Q_{im} T_{mn} Q_{jn}$
$= Q_{im} T_{mn} Q^t_{nj}$

2nd order tensors

$$T'_{ij} = Q_{im} Q_{jn} T_{mn} \quad : \quad = Q_{im} T_{mn} Q^t_{nj}$$

recall $A = BC$ $\quad A_{ij} = B_{im} C_{mj}$

$$[T'] = Q[T]Q^t$$

$$T'_{i_1 i_2 i_3} = Q_{i_1 j_1} Q_{i_2 j_2} Q_{i_3 j_3} T_{j_1 j_2 j_3}$$

n th order tensor

$$\boxed{\begin{aligned} T'_{i_1 \ldots i_n} &= \overbrace{Q_{i_1 j_1} \cdots Q_{i_n j_n}}^{n\ Q\text{'s}} T_{j_1 j_2 \ldots j_n} \\ T_{j_1 j_2 \ldots j_n} &= Q_{i_1 j_1} \cdots Q_{i_n j_n} T'_{i_1 \ldots i_n} \end{aligned}}$$