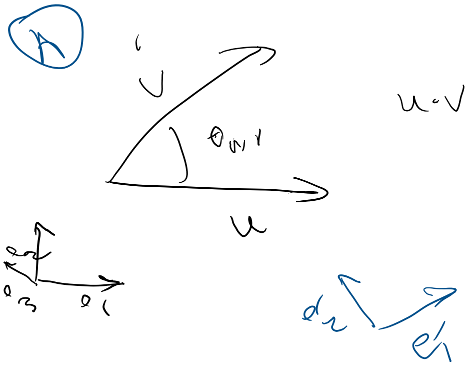
CM2023/09/13

Wednesday, September 13, 2023 9:43 AM

Inner product: Indicial representation

A



$u \cdot v = |u||v| \cos\theta_{u,v}$

I'll show that for an orthogonal basis

B $u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$

$= u_i v_i$

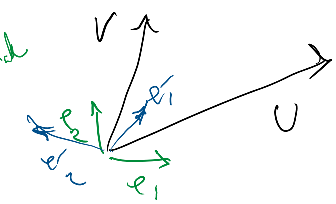
- Definition A is ideal in that it does not need a coordinate system so clearly that value of inner product number is INDEPENDENT of coordinate system => scalar
- Definition B uses a coordinate system and to show that the value of u.v is independent of coordinate system choice, we need to prove that it's value does not change from one coordinate system to another (not the best approach)
- Whenever possible, make the definitions independent of a coordinate system -> we automatically deal with tensors.

Assume we had used def B for inner product $\Longrightarrow$ need to prove that it's a scalar

? $u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$ for e coordinate system

$u \cdot v = u_1' v_1' + u_2' v_2' + u_3' v_3'$

are they equal



$u_i v_i$ coordinate system (e_1, e_2, e_3) $\overset{\text{we want to show}}{=}$ $u_i' v_i'$

$u_i = Q_{mi} u_m'$

$v_i = Q_{ni} v_n'$

$u_i v_i = (Q_{mi} u_m')(Q_{ni} v_n') = (Q_{mi} Q_{ni}) u_m' v_n'$

$\underbrace{(Q_{mi} Q^t_{in})} u_m' v_n' = (\underbrace{Q Q^t})_{mn} u_m' v_n'$

for orthonormal coordinate systems we use now $QQ^t = I$

$A = BC$

$A_{mn} = B_{mi} C_{in}$

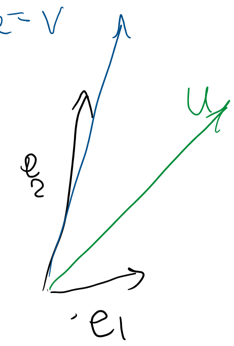
$\delta_{mn} u_m' v_n' = u_n' v_n' = u_i' v_i'$

$\delta_{mn} u_m u'_n = u_n u'_n = u_i v'_i$

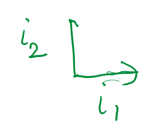
I proved $u_i v_i = u'_i v'_i$

BTW, why the value of inner product u.v is equal to ui vi?

$v_1 e_1 + v_2 e_2 = V$



orthogonal coordinate system



$u \cdot v = (u_1 e_1 + u_2 e_2) \cdot (v_1 e_1 + v_2 e_2)$

$= (u_1 e_1) \cdot (v_1 e_1) + (u_2 e_2) \cdot (v_1 e_1)$

$+ (u_1 e_1) \cdot (v_2 e_2) + (u_2 e_2) \cdot (v_2 e_2)$

$((a+b) \cdot c = a.c + b.c$

$c \cdot (a+b) =$

$u = u_1 e_1 + u_2 e_2$

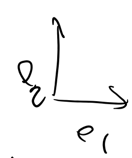
$= u_1 v_1 \, e_1 \cdot e_1 + u_2 v_1 \, e_2 \cdot e_1 +$

$u_1 v_2 \, e_1 e_2 + u_2 v_2 \, e_2 \cdot e_2$

$(\lambda a) \cdot (\mu b) = (\lambda \mu)(a \cdot b)$ Inner product homogeneity

possibilities

orthogonal coordinate system



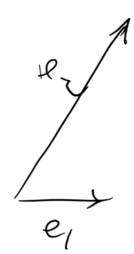
$e_i \cdot e_j = \delta_{ij}$

$u \cdot v = u_1 v_1 (1) + u_2 v_1 (0) + u_1 v_2 (0) + u_2 v_2 (1)$

$= u_1 v_1 + u_2 v_2 = u_i v_i$

$u \cdot v = u_i g_{ij} v_j$

General coordinate



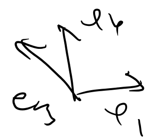
$g_{ij} = e_i \cdot e_j$

g is called metric matrix

Summary

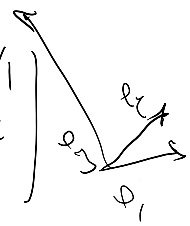
(1)

$$u \cdot v = u_i v_i = [v_1 \;\; u_2 v_3] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$



orthogonal

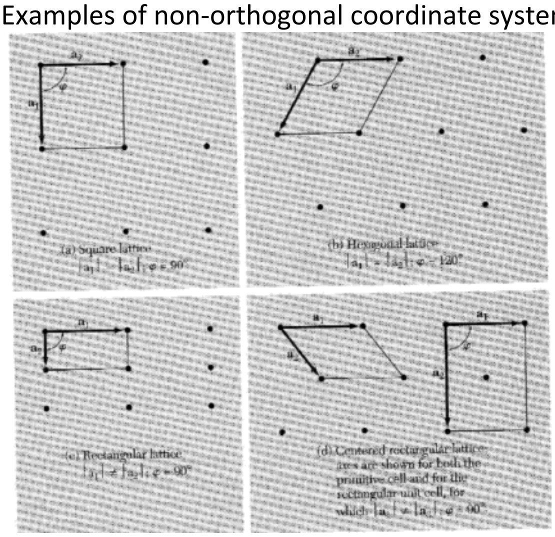
$$u \cdot v = u_i\, g_{ij}\, v_j = [u_1 \;\; u_2 \;\; u_3] \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & \cdot & - \\ - & - & g_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$



Non orthognal system

$$g_{ij} = e_i \cdot e_j$$

KittelCharles-IntroductionToSolidStatePhysics8ThEd.pdf
Examples of non-orthogonal coordinate systems for solid lattices



2nd order tensors = Linear operators

Assume that V and W are two vector spaces and the function L takes vectors from v in V to W

$$V \xrightarrow{L(v)} W$$

it must satisfy

(2)

1) $\forall v_1 \,\&\, v_2 \in V \quad L(v_1 + v_2) = L(v_1) + L(v_2)$

2) $\forall \lambda \in \mathbb{R} \,\&\, v_1 \in V \quad L(\lambda v_1) = \lambda L(v_1)$

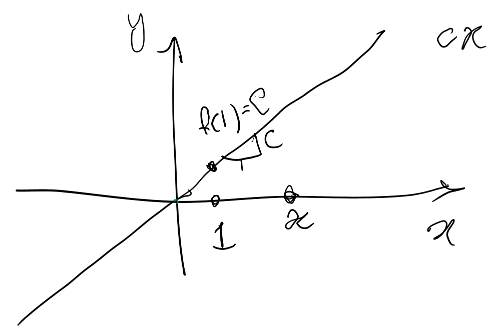
or in short $\quad L(\alpha u + v) = \alpha L(u) + L(v)$

Examples:

1D $\rightarrow$ 1D

$$f(x) = f(x \cdot 1) = x \underbrace{f(1)}_{\text{call this } c}$$

property 2 above



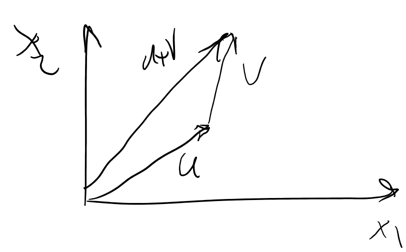
$$\boxed{f(x) = cx}$$

In general, linear functions pass through zero: $\quad L(0) = 0$

$$\cancel{0} + \cancel{L(0)} = L(0+0) = L(0) + \cancel{L(0)} \Rightarrow \boxed{L(0) = 0}$$

property 1 in def 2

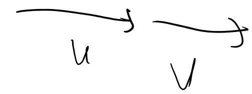
Example 2 $\quad \mathbb{R}^2 \rightarrow \mathbb{R}$



$$L(u) = |u|$$

def 2-1 $\quad L(u+v) \stackrel{?}{=} L(u) + L(v)$

$$|u+v| \le |u| + |v|$$



but not equal for all u & v

but not equal for all u & v

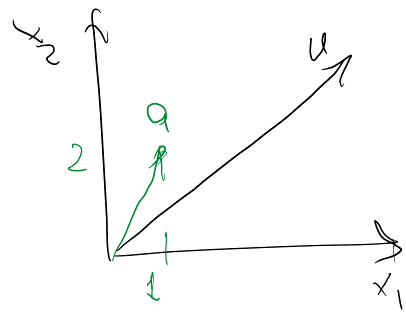
Not a linear function

Example 3

$\mathbb{R}^2 \longrightarrow \mathbb{R}$

$V \qquad W$

$a = (1,2)$



$a = (1,2)$

$$L_a(u) = a.u$$

I call this L of a as the function is defined based on a

Is L_a a linear function?

$$L_a(u) = a.u = a_1 u_1 + a_2 u_2 = u_1 + 2u_2$$ (4)

I need to show $L_a(\alpha u + v) = \alpha L_a(u) + L_a(v)$

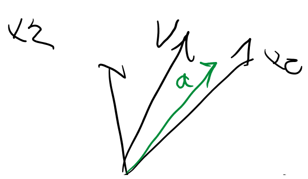
let's check this

$$L_a(\alpha u + v) = (\alpha u + v)_1 + 2(\alpha u + v)_2$$
$$= \alpha u_1 + v_1 + 2\alpha u_2 + 2v_2$$
$$= \alpha(u_1 + 2u_2) + (v_1 + 2v_2)$$ (4) again
$$= \alpha L_a(u) + L_a(v)$$

In fact, any linear function from R^d to R, can be expressed as an inner product

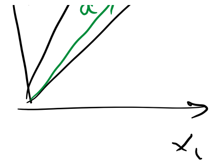
d dimensional Euclidean vector space

(5)



$L(u)$ is linear $R^3 \to R$ $\Longrightarrow$ there is a vector a $\exists$ such that

$$L(u) = a \cdot u$$



$L(u) = a \circ u$

There is one remaining point to consider before proceeding on to second-order tensors. That is, the inner product allows us to interpret a (Euclidean) vector as a linear operator that maps a vector into a real number (scalar). In fact, this is the defining property of a *first-order tensor*, and vectors are indeed first order tensors. To illuminate this point, let $\mathcal{V}$ be the set of all vectors in some Euclidean point space $\mathcal{E}$. Now consider a specific vector $\bar{\mathrm{a}} \in \mathcal{V}$, where the overbar indicates that we hold $\bar{\mathrm{a}}$ fixed. We can define a function $f_{\bar{\mathbf{a}}}$ that maps a vector into a scalar by taking the inner product of $\bar{\mathrm{a}}$ and any vector $\mathbf{b} \in \mathcal{V}$. That is,

$$f_{\bar{\mathbf{a}}}(\mathbf{b}) \equiv \bar{\mathrm{a}} \cdot \mathbf{b}.$$

A review of the properties of the inner product shows that $f_{\bar{\mathbf{a}}}$ is indeed a linear operator. In fact, the *Riesz representation theorem* states that *every* linear function on $\mathcal{V}$ to $\Re$ can be represented in this fashion (by varying our choice of the fixed vector $\bar{\mathrm{a}}$)! We use a similar approach in the next section to define second-order tensors as a special class of linear operators.

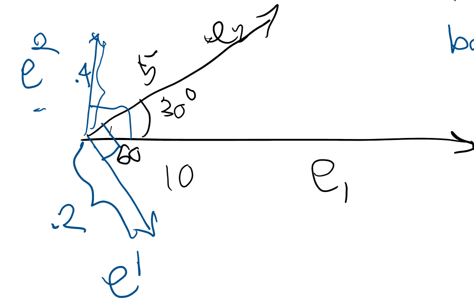
FYI: non-orthogonal coordinate systems and correct way of indexing

$L: \mathbb{R}^d \to \mathbb{R}$ there is a $a \ni L(u) = u \cdot a$

linear $d = 3$ (3D)

$e^i \ni e^i(e_j) = \delta^i_j$

$\{e_1, e_2\}$ actual basis



linear function from $\mathbb{R}^2 \to \mathbb{R}$

$e^1(e_1) = 1$ $e^1(e_2) = 0$ from (5)

$e^2(e_1) = 0$ $e^2(e_2) = 0$

$e^1 \cdot e_1 = 1$ $e^1 \cdot e_2 = 0$

$e^2 \cdot e_1 = 0$ $e^2 \cdot e_2 = 0$

Vectors covectors

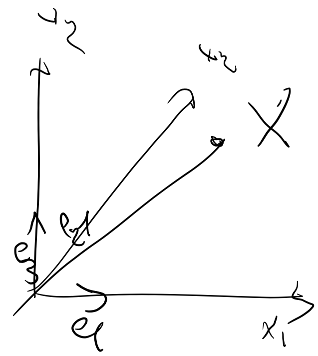
Vectors

velocity $v = v^i e_i$

displacement $u = u^i e_i$

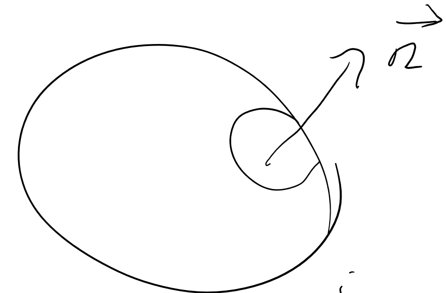
position $X = X^i e_i$

components of vector X



Covectors
basis for that is
(e^1, e^2, e^3)

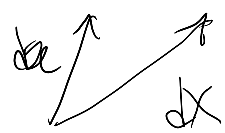
covectors in mechanics



$$n = n_i e^i$$

for example
Chapter 2

$$F$$



dX is displaced

$$dx = F \, dX$$

vector   vector

$$dx^i = F^i{}_j \, dX^j$$

another example

stress tensor
chapter 3

$$\underset{\text{vector}}{\overset{\text{traction}}{t}} = \overset{\text{stress}}{\sigma} \; \underset{\text{covector}}{n}$$

$$t^i = \sigma^{ij} n_j$$

Back to tensor

From this point on, we are not going to be careful with up and down indices (not a big a deal for orthogonal coordinate systems)

Linear functions form a vector space themselves

if S, T, U are linear functions from space $U \to W$

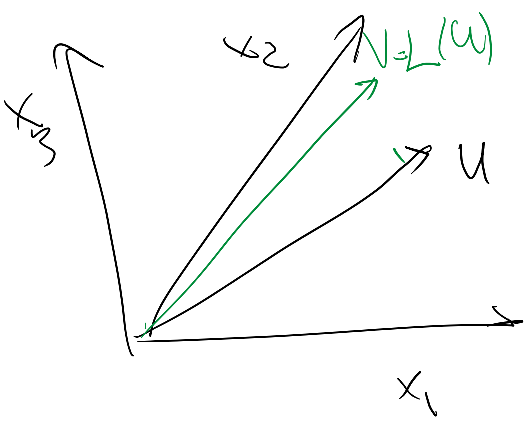
$$\begin{cases} S+T = T+S \\ S+(T+U) = (S+T)+U \\ S+0 = S \end{cases}$$

$$\begin{cases} (\lambda\mu)S = \lambda(\mu S) \\ \lambda(S+T) = \lambda S + \lambda T \\ (\lambda+\mu)S = \lambda S + \mu S \\ 1S = S \end{cases}$$

$(S+T)(u) := S(u) + T(u)$

$(\lambda S)(u) := \lambda S(u)$

A linear operator from vectors to vectors



$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \underbrace{\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix}}_{\text{matrix}} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

map from vectors to vectors

A second order tensor is basically a linear map that takes vector to vectors and in general higher order tensors map tensors to tensors

$$\underset{\text{2nd}}{\sigma} = \underset{\text{4th}}{C}\ \underset{\text{2nd order}}{\varepsilon}$$

2nd order tensor

4th order tensor

2nd order tensor