CM2023/09/18

Monday, September 18, 2023 9:19 AM

Dyadic products

Recall I loosely defined

inner product $$u\cdot v = [u_1 \cdots u_n]\begin{bmatrix}v_1\\ v_2\\ \vdots\\ v_n\end{bmatrix} = u^t v = u_i v_i \qquad u = \begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix} \quad v = \begin{bmatrix}v_1\\ \vdots\\ v_n\end{bmatrix}$$

$$u\otimes v = \begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix}[v_1 \cdots v_n] = uv^t = \begin{bmatrix} u_1v_1 & u_1v_2 & \cdots & u_1v_n\\ u_2v_1 & \cdots & & \\ & & & \\ u_nv_1 & & & u_nv_n\end{bmatrix}$$

Definition of dyadic product should be coordinate-independent

for vectors u & $v \in \mathcal{V}$ we define the dyadic product as an operator from $\underline{\mathcal{V} \to \mathcal{V}}$:

$$\forall w\in\mathcal{V} \qquad \underbrace{(u\otimes v)}\,\underset{\text{vector}}{\underset{\downarrow}{w}} = \overbrace{\underset{\text{vector}}{\underset{\downarrow}{u}}\,(\underbrace{v\circ w}_{\substack{\text{inner product}\\ b\\ \text{scalar}}})}^{\text{vector}} \qquad ①$$

first, I show $(u\otimes v)$ is a linear operator:

want to show

$$(u\otimes v)(a+\beta b) = (u\otimes v)a + \beta\,(u\otimes v)b$$

Proof:

$$(u\otimes v)(a+\beta b) = u\,(v\circ(a+\beta b)) \qquad \text{def } ①$$

$$= u(v\cdot a + \beta(v\cdot b)) \qquad \text{dist \& homogeneity property of inner product}$$

$$= \underbrace{u(v\cdot a)}_{\text{def } ①} + \beta\,\underbrace{u(v\cdot b)}_{\downarrow}$$

dist: property of addition in vector space $u(\lambda+\mu) = u\lambda + u\mu$

$$= (u\otimes v)a + \beta\,(u\otimes v)b$$

So dyadic product is a linear operator from vectors $\mathcal{V}$ to vectors $\mathcal{V}$

u vector
v vector $\Longrightarrow$ $u \otimes v$ is a 2nd order tensor

we'll show its component representation is

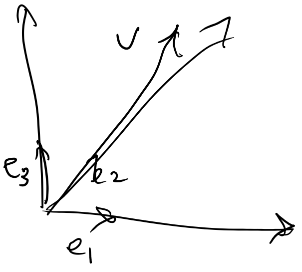
$$u \otimes v = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} [v_1 \cdots v_n] = uv^t = \begin{bmatrix} u_1v_1 & u_1v_2 \cdots u_1v_n \\ u_2v_1 \cdots & \frown \\ u_nv_1 & & u_nv_n \end{bmatrix}$$

Motivation on what the basis is for 2nd order tensors:

$V = v^1e_1 + v^2e_2 + v^3e_3$ $V = (v^1, v^2, v^3)$

$$e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} \quad e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} \quad e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$v = \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = v_1\underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}}_{e_1} + v_2\underbrace{\begin{pmatrix}0\\1\\0\end{pmatrix}}_{e_1} + v_3\underbrace{\begin{pmatrix}0\\0\\1\end{pmatrix}}_{e_3}$$



2nd order tensor expressed in e coordinate

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} = T_{11}\overset{e_1\otimes e_1}{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}} + T_{12}\underbrace{\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}}_{e_1\otimes e_2} + T_{13}\underbrace{\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}}_{e_1\otimes e_3}$$

$$+ \cdots + T_{33}\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

$$= T_{ij}\, e_i \otimes e_j$$

basis for 2nd order tensors

$$e_1 \otimes e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}[1 \quad 0 \quad 0] = \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$$e_1 \otimes e_2 = \begin{bmatrix}1\\0\\0\end{bmatrix}[0 \quad 1 \quad 0] = \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}$$

$$e_2 \otimes e_1 = \begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix} \qquad e_2 \otimes e_1 = (e_1 \otimes e_2)^t$$

Basically the same way for vectors we had

$V = V_i e_i$ $v_i = V \cdot e_i$ in an orthogonal coordinate system we have

$$T = T_{ij}\, e_i \otimes e_j \qquad \text{where} \qquad T_{ij} = \underbrace{\underbrace{e_i}_{\text{vector}} \cdot \underbrace{(Te_j)}_{\text{vector}}}_{\text{scalar}} \quad ②$$

• inner product

$T_{12} = e_1 \cdot Te_2$

$T_{21} = e_2 \cdot Te_1$

proof

we want to show $\forall v \in V$ we have

$$\underbrace{Tv = (T_{ij}\, e_i \otimes e_j)\, v}_{} \implies T = T_{ij}\, e_i \otimes e_j$$

why this is correct?

$$Tv = (Tv)_i\, e_i$$

$$= (e_i \cdot Tv)\, e_i$$

$$= (e_i \cdot \underbrace{T(v_j e_j)}_{})\, e_i$$

$$= (v_j\, \underbrace{(e_i \cdot Te_j)}_{\text{def ②}})\, e_i$$

$$= (v_j\, T_{ij})\, e_i$$

$$= T_{ij}\, (v_j e_i)$$

$$= T_{ij}\, ((e_i \otimes e_j)\, v)$$

$$= (T_{ij}\, e_i \otimes e_j)\, v$$

why $u = u_i e_i$ $(u = Tv)$

$\downarrow$

$u_i = e_i \cdot u$ $((Tv)_i = (e_i \cdot Tv))$

$v = v_j e_j$

using this $T(\alpha a + \beta b) = \alpha T(a) + \beta T(b)$

linearity of T

def of dyadic product

$(e_i \otimes e_j)\, v = e_i\, (e_j \cdot v) = e_i\, v_j$

basically I showed $Tv = (T_{ij}\, e_i \otimes e_j)\, v \quad \forall v$

so

$$\boxed{T = T_{ij}\, e_i \otimes e_j \qquad T_{ij} = e_i \cdot Te_j}$$

③

$$T = T_{ij}\, e_i \otimes e_j \qquad T_{ij} = e_i \cdot T e_j$$

or in expanded form

$$T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = T_{11} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{e_1 \otimes e_1} + T_{12} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{e_1 \otimes e_2} + \cdots + T_{33} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{e_3 \otimes e_3}$$

Theorem 51 *Let* $\mathbf{T} \in \operatorname{Lin} \mathcal{V}$ *with components* T_{ij} *w.r.t. the r.C.c.f.* X*, and let* $\mathbf{u}, \mathbf{v} \in \mathcal{V}$ *with components* u_i, v_i *w.r.t.* X*. Then*

$$\boxed{\mathbf{Tu} = \mathbf{v}} \Leftrightarrow \underbrace{T_{ij}u_j = v_i}.$$

Proof. *Suppose* $\mathbf{Tu} = \mathbf{v}$*. Then*

$$\begin{aligned}
v_i &= \mathbf{e}_i \cdot \mathbf{v} \ \textit{(components of } \mathbf{v}\textit{)}, \\
&= \mathbf{e}_i \cdot (\mathbf{Tu}) \ \textit{(supposition)}, \\
&= \mathbf{e}_i \cdot [(T_{kl}\mathbf{e}_k \otimes \mathbf{e}_l)\,\mathbf{u}] \ \textit{(component representation of } \mathbf{T}\textit{)}, \\
&= \mathbf{e}_i \cdot [T_{kl}\,(\mathbf{e}_k \otimes \mathbf{e}_l)\,\mathbf{u}] \ \textit{(defns. of tensor + and scalar mult.)}, \\
&= \mathbf{e}_i \cdot [T_{kl}\,(\mathbf{e}_l \cdot \mathbf{u})\,\mathbf{e}_k] \ \textit{(defn. of } \otimes \textit{)}, \\
&= \mathbf{e}_i \cdot [T_{kl}\,(u_l\mathbf{e}_k)] \ \textit{(components of } \mathbf{u}\textit{)}, \\
&= T_{kl}u_l\,(\mathbf{e}_i \cdot \mathbf{e}_k) \ \textit{(distr. and homog. of } \cdot \textit{)}, \\
&= T_{kl}u_l\delta_{ik} \ \textit{(orthonormality of base vectors)}, \\
&= T_{il}u_l = T_{ij}u_j \ \textit{(property of } \delta_{ik} \textit{ and labeling)}. \\
\therefore \mathbf{Tu} &= \mathbf{v} \Rightarrow T_{ij}u_j = v_i.
\end{aligned}$$

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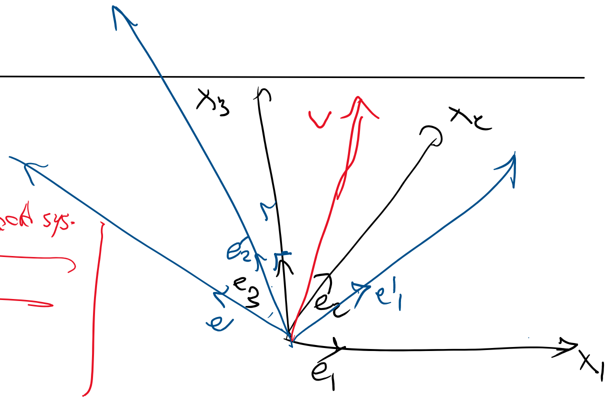
e & e' are orthonormal coordinate systems



$$v = v_i e_i = v'_i e'_i$$

$$v'_i = Q_{ij}\, v_j \qquad Q: \begin{bmatrix} e'_1 \text{ in e coord sys.} \\ e'_2 \\ e'_3 \end{bmatrix}$$

$$[v'] = Q[v]$$

we mentioned for a 2nd order tensor

$$T'_{mn} = Q_{mi}\, Q_{nj}\, T_{ij} = \underbrace{Q_{mi} T_{ij}}\underbrace{Q^t_{jn}} = (QTQ^t)_{mn}$$

$$[T'] = Q[T]Q'$$

④

why ④ is true

Intuitive. demonstrate:

T: 2nd order tensor

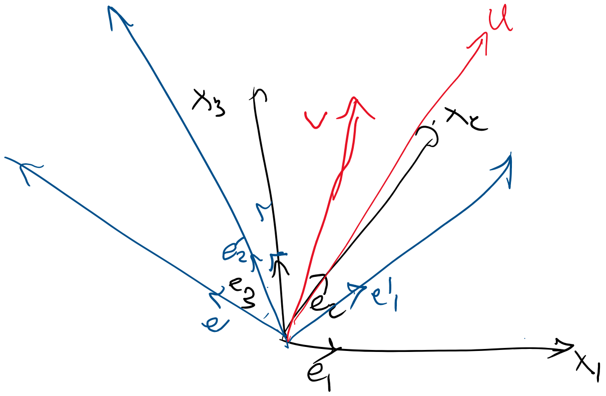
why ④ is true

Intuitive demonstration:

T: 2nd order tensor

$Tu = v$

we want to obtain components of T in e' coordinate system:



i) $[v]' = Q[v] \Longrightarrow [v] = Q^t[v]'$

components of v in e' system

components of v in e coordinate system

ii) $[u]' = Q[u]$

want to create $\underline{[v]'} = \overbrace{(\qquad)}^{\text{components of T in e' system}}\underline{[u]'}$

$[v] = [T][u]$ because $v = Tu$

components of T in e system

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & & \\ T_{31} & - & T_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

i $\downarrow$

$Q^t[v]' = [T](Q^t[u]')$ from ii $\Longrightarrow Q^t[v]' = [T]Q^t[u']$

premultiply by Q

$$\underbrace{QQ^t}_{Q \text{ is orthonormal}}[v]' = (Q[T]Q^t)[u]'$$

$$[v]' = \underbrace{(Q[T]Q^t)}_{[T]'}[u]'$$

$$[T]' = Q[T]Q^t \qquad \begin{bmatrix} T'_{11} & T'_{12} & T'_{13} \\ T'_{21} & T'_{22} & T'_{23} \\ T'_{31} & T'_{32} & T'_{33} \end{bmatrix} = Q \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} Q^t$$

④ same as above

$$\Longrightarrow T'_{mn} = Q_{mi}T_{ij}Q^t_{jn} = Q_{mi}Q_{nj}T_{ij}$$

A simpler but less intuitive proof of this:

$T = T_{ij}\, e_i \otimes e_j$ want to obtain $T = T'_{mn}\, e'_m \otimes e'_n$

$e_i = Q_{mi}e'_m$

$e_j = Q_{nj}e'_n$

$\Longrightarrow T = T_{ij}(Q_{mi}e'_m) \otimes (Q_{nj}e'_n)$

becau

$$\left.\begin{aligned} e_i &= Q_{mi} e'_m \\ e_j &= Q_{nj} e'_j \end{aligned}\right\} \Rightarrow T = T_{ij}(Q_{mi} e'_m) \otimes (Q_{nj} e'_n)$$

$$= \underbrace{Q_{mi} Q_{nj} T_{ij}}_{T_{mn}} e'_m \otimes e'_n$$

because $\otimes$ is a linear operator

Theorem 53| *Let there be a set of nine real numbers associated with every r.C.c.f. in a three-dimensional Euclidean point space $\mathcal{E}$. For example, consider the sets $\{X, \mathbf{e}_i, T_{ij}\}$ and $\{X', \mathbf{e}'_i, T'_{ij}\}$ where X and X' are arbitrary frames with respective base vectors $\mathbf{e}_i$ and $\mathbf{e}'_i$. Then $\exists$ ("there exists") $\mathbf{T} \in$ Lin $\mathcal{V}$ given by $\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = T'_{ij}\mathbf{e}'_i \otimes \mathbf{e}'_j$ iff the two-index symbols T_{ij} and T'_{ij} satisfy the transformation rules*

$$T'_{ij} = \overset{Q}{\lambda_{ik}}\lambda_{jl}T_{kl}; \quad T_{ij} = \lambda_{ki}\lambda_{lj}T'_{kl}.$$

for all choices of $\{X, \mathbf{e}_i, T_{ij}\}$ and $\{X', \mathbf{e}'_i, T'_{ij}\}$, where λ_{ij} are the cosines of the angles between $\mathbf{e}'_i$ and $\mathbf{e}_j$.

Other definitions related to 2nd order tensors
Transpose of a 2nd order tensor

T: is a 2nd order tensor

T^t: " " " " " defined as

$\forall u, v \in V \quad v \circ T^t u = u \circ T v$

consequence of this

$$T_{ij} = \underset{\underset{u}{\downarrow}}{e_i} \circ T \underset{\underset{v}{\downarrow}}{e_j} = e_j \circ T^t e_i = T^t_{ji}$$

def

some properties of transpose

1) $(T^t)^t = T$

2) $(T+S)^t = T^t + S^t$

3) $(\alpha T)^t = \alpha T^t$

4) $I^t = I$

3)

4) $I^t = I$

5) $(u\otimes v)^t = v\otimes u$

Hint $e_1\otimes e_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$e_2\otimes e_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$T_{ij} = e_i \cdot T e_j$

$$\Big(\underbrace{(u\otimes v)^t}_{T}\Big)_{ij} \quad e_i \cdot (u\otimes v)^t e_j$$

$$= e_j \cdot \big((u\otimes v) e_i \big) = e_j \cdot \big(u\,(v\cdot e_i)\big) = (e_j\cdot u)\,(v\cdot e_i) =$$

$$\underline{u_j v_i} = e_i \big((v\otimes u) e_j \big)$$

Some definitions of tensor spaces

$$Sym = \{ T \in Lin \mid T = T^t \}$$

symmetric matrices

2nd order tensors

examples stress & strain tensors

$$Skew = \{ T \in Lin \mid T^t = -T \}$$

we'll see rotation as an example

$$T = \underbrace{\frac{T+T^t}{2}}_{sym\,T} + \underbrace{\frac{T-T^t}{2}}_{skew\,T}$$

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det T

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$