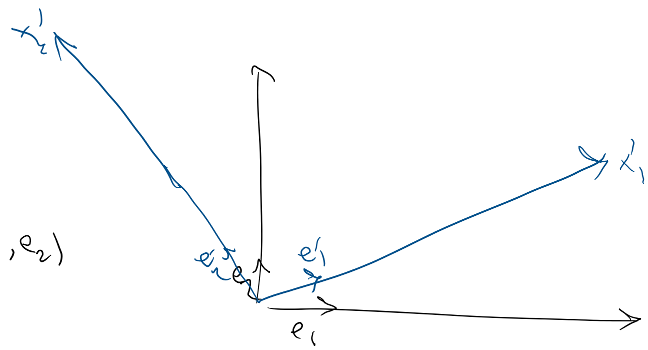
CM2023/09/25

Monday, September 25, 2023 9:47 AM



Determinant of a tensor

T_{ij} a 2nd order tensor

$[T_{ij}]$ expression of tensor T in (e_1, e_2) coordinate system.

$T = T_{ij} e_i \otimes e_j$

$$\det T := \det \underbrace{[T_{ij}]}_{\text{matrix: components of } T \text{ in } (e) \text{ coordinate system}}$$

Now that we have used a coordinate system to define det T, we need to prove that it's value is not going to change if we use another orthonormal coordinate system:

I want to $\det [T'_{ij}] = \det [T_{ij}]$

$$\det [T'_{ij}] = \frac{1}{6} \epsilon_{ijk} \epsilon_{pqr} T'_{ip} T'_{jq} T'_{kr}$$

$$T'_{ip} = Q_{ia} Q_{pd} T_{ad}$$

$$T'_{jq} = Q_{jb} Q_{qe} T_{be}$$

$$T'_{kr} = Q_{kc} Q_{rf} T_{cf}$$

$$\det [T'] = \frac{1}{6} \epsilon_{ijk} \epsilon_{pqr} (Q_{ia} Q_{pd} T_{ad})(Q_{jb} Q_{qe} T_{be})(Q_{kc} Q_{rf} T_{cf})$$

$$= \frac{1}{6} \underbrace{(\epsilon_{ijk} Q_{ia} Q_{jb} Q_{kc})}_{(\epsilon_{abc} \det Q)} \underbrace{(\epsilon_{pqr} Q_{pd} Q_{qe} Q_{rf})}_{(\epsilon_{def} \det Q)} T_{ad} T_{be} T_{cf}$$

$\det Q = 1$ (Q is orthonormal)

$$= \frac{1}{6} \epsilon_{abc} \epsilon_{def} T_{ad} T_{be} T_{cf} = \det [T]$$

So, determinant is an invariant of a 2nd order tensor (its value does not change by the choice of orthonormal coordinate system).

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coordinate system).

Trace

Good definition (coordinate system independent)

We do two things to define trace for an arbitrary second order tensor:

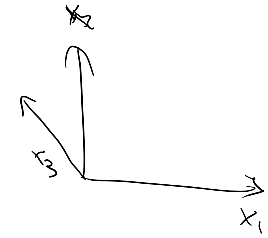
1. Trace is linear

$$tr(\alpha S + \beta T) = \alpha\, tr(S) + \beta\, Tr(T)$$

2. Define traces for "building blocks" of second order tensors

Bad definition

$$T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$



$$\text{trace}([T]) = T_{11} + T_{22} + T_{33} = T_{ii}$$

Not a good def, because I've use a coordinate system for this!

$$tr(u \otimes v) = u \cdot v$$

$$u \otimes v = \begin{bmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{bmatrix}$$

$$u \cdot v = u_1v_1 + u_2v_2 + u_3v_3$$

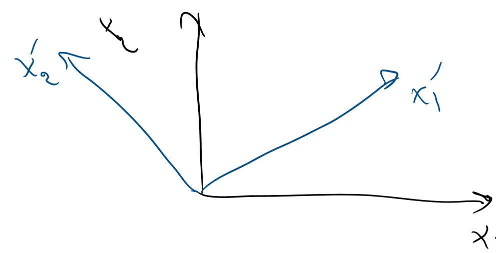
Use properties 1 & 2 to show $tr(T) = T_{ii}$

$$tr(T) = tr(T_{ij}\, e_i \otimes e_j) \underset{(1)\ \text{linearity of } tr}{=} T_{ij}\, tr(e_i \otimes e_j) \overset{(2)}{=} T_{ij}\, e_i \cdot e_j$$

$$= T_{ij}\,\delta_{ij} = T_{ii}$$

Do I need to show that Tii = T'ii

$$\underbrace{T_{ii}}_{tr(T)} \overset{?}{=} \underbrace{T'_{ii}}_{tr(T)}$$



We don't need to prove this as both sides are simply equal to the coordinate-independent trace definition.

Still, if we want to check this (not necessary)

$$T'_{mn} = Q_{mi}\,Q_{nj}\,T_{ij} \qquad \left(\sum_{m=1}^{3} T'_{mm} = Q_{mi}\,Q_{mj}\,T_{ij}\right) \quad n \to m \quad \text{to create trace}$$

$$T'_{mm} = Q_{mi}\,Q_{mj}\,T_{ij} = \underbrace{Q^t_{im}\,Q_{mj}}\,T_{ij} = \underbrace{(Q^tQ)_{ij}}_{I_{ij} = \delta_{ij}}\,T_{ij}$$

$$= \delta_{ij}\,T_{ij} = T_{ii}$$

Properties of trace:

1. $\mathrm{tr}(T) = \mathrm{tr}(T^t)$
2. $\mathrm{tr}(ST) = \mathrm{tr}(TS)$
3. $\mathrm{tr}(I_d) = d$ $\quad d=2 \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
4. $\mathrm{tr}(0) = 0$

Inner product and norm for a second order tensor:
There are many definitions for <.,.> and norm for 2nd order tensors:

Def 1 of a norm that comes out of an inner product

$$S : T = \mathrm{tr}(ST^t)$$

$$\mathrm{tr}(ST^t) = (ST^t)_{ii}$$

Note $(ST^t)_{ij} = \underbrace{S_{ik} (T^t)_{kj}}$ $\Rightarrow$ $\boxed{(ST^t)_{ij} = S_{ik} T_{jk}}$

$j \to i$

$$= S_{ik} T_{\underline{i}k} = S_{11}T_{11} + S_{12}T_{12} \cdots + S_{33}T_{33}$$

$j \to i$

$$S : T = \mathrm{tr}(ST^t) = S_{ik}T_{ik} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} : \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

or $S : T$

$$= S_{11}T_{11} + S_{12}T_{12} + S_{13}T_{13} + \cdots + S_{33}T_{33}$$

Comment: this looks like vector inner product

$$U.V = U_i V_i$$

$$\|T\| = \sqrt{T : T}$$

T: measured stress
S: exact stress

$$\underset{\text{error}}{\Delta} = S - T$$

stress error $= \|\Delta\| = \sqrt{\Delta : \Delta} = \sqrt{(S-T):(S-T)}$

$$\text{stress error} = \|\Delta\| = \sqrt{\Delta \cdot \Delta} = \sqrt{(S - T) : (S - T)}$$
$$= (S_{ij} - T_{ij})\,(S_{ij} - T_{ij})$$

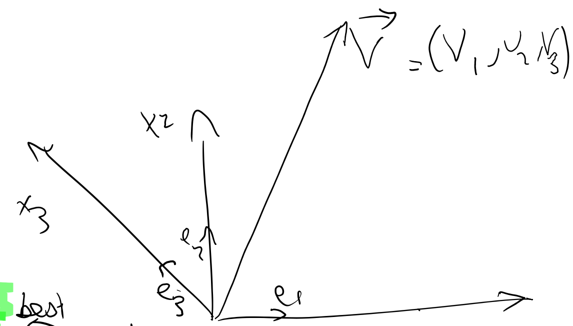
There is potentially a better (but more difficult to calculate) norm for second order tensors:

Vector-induced norm of a second order tensor:

T 2nd order tensor $\qquad \mathcal{V} \longrightarrow \mathcal{V}$ maps vectors to vectors

Vectors themselves already have a norm:

For example, for Euclidian vectors we had various definitions of norm:



$$\|V\|_p = \sqrt[p]{|V_1|^p + |V_2|^p + |V_3|^p}$$

Examples

$$\|V\|_2 = \sqrt{|V_1|^2 + |V_2|^2 + |V_3|^2}$$ best ① coordinate indep. ② it comes from $\cdot$: $a \cdot b = a_i b_i$

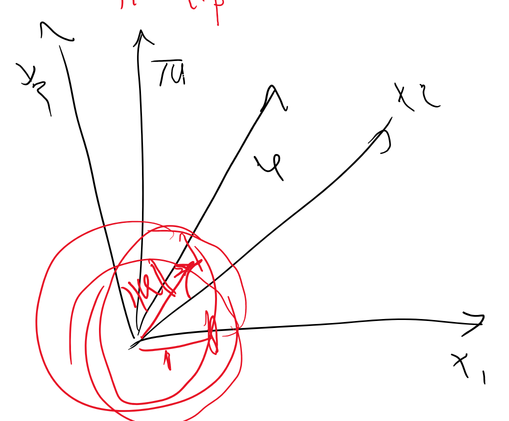
$$\|V\|_1 = |V_1| + |V_2| + |V_3|$$

$$\|V\|_\infty = \mathrm{Max}\,(|V_1|, |V_2|, |V_3|) \qquad (p \to \infty)$$

Now that we have a norm for vectors we can define a norm for 2nd order tensors:

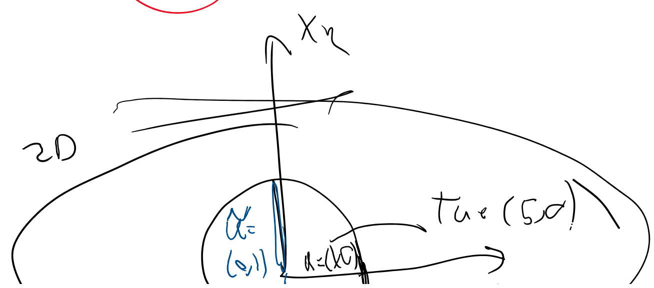
$\| \;\; \|^V$: vector norm like $\| \;\; \|_p$ above

$$\boxed{\begin{gathered}\|T\| = \underset{u \neq 0}{\mathrm{Max}} \frac{\|Tu\|^V}{\|u\|^V} \\ \Rightarrow \\ \|T\| = \underset{\|u\|^V = 1}{\mathrm{Max}} \|Tu\|^V\end{gathered}}$$



Example

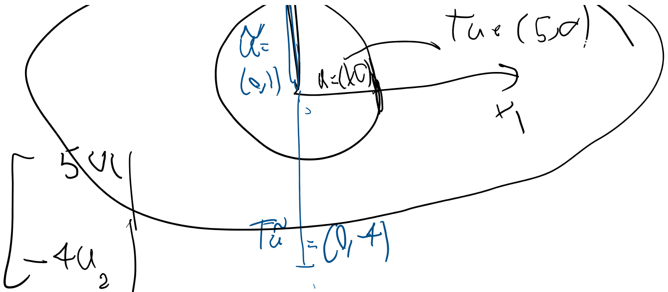
$$T = \begin{bmatrix} 5 & 0 \\ 0 & -4 \end{bmatrix}$$



$$= \begin{bmatrix} \cdot & \cdot \\ 0 & -4 \end{bmatrix}$$

$\|T\| = ?$

$$\begin{bmatrix} 5 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 5v_1 \\ -4u_2 \end{bmatrix}$$



$\|T\|_2 = 5$

$$\|T\|_2 = \max_{u \neq 0} \frac{\|Tu\|_2}{\|u\|_2} = \max_{\|u\|_2 = 1} \|Tu\|_2 =$$

$$\rho(T) = \max(|\lambda_i|)$$

ρ: spectral density of T

λ_i: eigenvalues of T

We extend this "vector-induced" norm even to higher even-order tensors

$$\sigma = C \; \varepsilon$$

σ: 2nd order stress tensor

C: 4th order elasticity tensor

ε: 2nd order strain tensor

linear elasticity

$$\|C\| = \max_{\varepsilon \neq 0} \frac{\|C\varepsilon\|}{\|\varepsilon\|} = \max_{\|\varepsilon\| = 1} \|C\varepsilon\|$$

?

$$\max_{u\neq 0} \frac{\|Tu\|}{\|u\|} \overset{?}{=} \max_{\|a\|=1} \|Ta\|$$

$$\max_{u\neq 0} \left\| \frac{1}{\|u\|} T u \right\| = \max_{u\neq 0} \left\| T\left(\frac{u}{\|u\|}\right)\right\|$$

call this a

$$\max_{\|a\|=1} \|Ta\|$$

Inverse of a second order tensor:

$$T T^{-1} = T^{-1} T = I$$

identity tensor

Theorem 76: the components of the inverse of a 2nd order tensor are given as:

$$\boxed{T^{-1}_{ij} = \frac{1}{2\det T}\, \epsilon_{ipq}\, \epsilon_{jmn}\, T_{mp}\, T_{nq}}$$

Inverse exists only when det T $\neq 0$

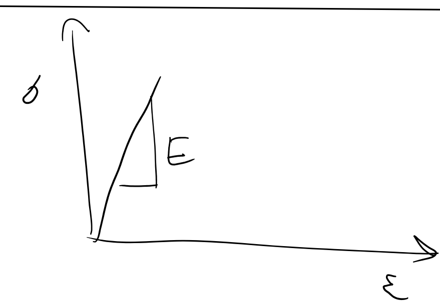
invertible tensors

$$\mathrm{Inv}\, V = \{ T \in \mathrm{Lin}\, V \mid \det T \neq 0 \}$$

$$\mathrm{Lin}\, V^{+} = \{ T \in \mathrm{Lin}\, V \mid \det T > 0 \}$$

Higher order tensors
Motivation:
In 1D stress is related to strain through Elastic modulus E

$$\sigma = E\varepsilon \qquad \varepsilon = u_{,x}$$



In 2D and 3D, stress and strain are both second order tensors

u

ε

In 2D and 3D, stress and strain are both second order tensors

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & - & \\ & - & \sigma_{33} \end{bmatrix} \qquad \varepsilon = \left(\frac{\nabla u + \nabla u^{t}}{2} \right) \qquad \varepsilon = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & - & \\ & - & \varepsilon_{33} \end{bmatrix}$$

$$\sigma = \underbrace{\begin{bmatrix} \sigma_{11} & - & \sigma_{13} \\ - & & \\ & - & \sigma_{33} \end{bmatrix}}_{\text{2 indices}} = C \underbrace{\begin{bmatrix} \epsilon_{11} & - & \epsilon_{13} \\ & & \\ - & & \epsilon_{33} \end{bmatrix}}_{\text{2 indices}}$$

indicial notation

$$\sigma_{ij} = C_{ijlk}\, \epsilon_{kl}$$

4th order Elasticity tensor