$$\mathcal{E} = \varepsilon_{ijk}\, e_i \otimes e_j \otimes e_k$$

permutation 3rd order tensor

Polyads are the generalization of dyadic products:

(P) $$\underbrace{(u_1 \otimes u_2 \otimes u_3 \otimes \cdots \boxed{u_n})}_{n^{th}\text{ order tensor}} \boxed{\underset{\text{vector}}{w}} = \underbrace{(u_1 \otimes u_2 \otimes \cdots \otimes u_{n-1})}_{n-1 \text{ order tensor}} \overset{\text{scalar value}}{(u_n \cdot w)}$$

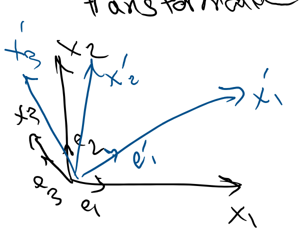
n-1 order tensor

generalization of

$$\underbrace{(u_1 \otimes \boxed{u_2)\, w}}_{2nd\text{ order tensor}} = u_1\, \boxed{(u_2 \cdot w)}$$

Components of a tensor:

order	1 (vector)	2 (2nd order tensor)	m (m^{th} order tensor)
	$V = V_i e_i$	$T = T_{ij}\, e_i \otimes e_j$	$T = T_{i_1 \cdots i_m}\, e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_m}$
formulas for components (orthonormal coordinate system)	$V_i = V \cdot e_i$	$T_{ij} = e_i \cdot (T e_j)$	(*) $T_{i_1 \cdots i_m} = e_{i_1} \cdot \Big(\underbrace{\underbrace{\big(\underbrace{(T e_{i_m})}_{m-1 \text{ order tensor}} e_{i_{m-1}}\big)}_{m-2 \text{ order}} \cdots}_{\text{vector}} \Big) e_{i_2}$
coordinate transformation	$V'_i = Q_{ij} V_j$	$T'_{ij} = Q_{im} Q_{jn} T_{mn}$	(**) $T'_{i_1 \cdots i_m} = Q_{i_1 j_1} Q_{i_2 j_2} \cdots Q_{i_m j_m} T_{j_1 \cdots j_m}$



(2)

See TAM551:

- Def 46 for components of an m'th order tensor
- Theorem 84 for equation (*)
- Theorem 88 for coordinate transformation of an m'th order tensor (**)

Tensor product in general:

$\overset{m}{T}$ $\qquad$ m^{th} order tensor

$\overset{n}{S}$ $\qquad$ n^{th} order tensor

$$\underbrace{\left(\overset{m}{T}\,\overset{n}{S}\right)}_{\text{tensor product}} = m-n \quad \text{order tensor}$$

$$\left(\overset{m}{T}\,\overset{n}{S}\right)_{i_1 \cdots i_{m-n}} = \overset{m}{T}_{i_1 \cdots i_{m-n}\, i_{m-n+1} \cdots i_m}\ \overset{n}{S}_{i_m\, i_{m-1}\, i_{m-2} \cdots i_{m-n+1}}$$

C $\quad$ 4th order

ε $\quad$ 2nd order

$$\underset{\text{2nd order}}{\sigma} = \underset{4}{C}\ \underset{2}{\varepsilon} \qquad 2 = 4 - 2$$

$$\sigma_{i_1 i_2} = C_{i_1 i_2 i_3 i_4}\, \varepsilon_{i_4 i_3}$$

$$\sigma_{ij} = C_{ijkl}\, \varepsilon_{lk}$$ ✓ because ε is sym

③ $\varepsilon_{lk} = \varepsilon_{kl} \rightarrow \sigma_{ij} = C_{ijkl}\, \varepsilon_{kl}$

Other examples

$$\underset{\text{vector}}{v} = \underset{\text{2nd order}}{T}\ \underset{\text{1st order}}{u}$$

$$1 = 2 - 1$$

$(T:S^t) = (T \odot S) = T_{ij} \; S_{ji} = T_{ij} S^t_{ij} = T : S^t$

4a

$T_{ij} \downarrow$ 2nd order, $S_{ji} \downarrow$ 2nd order

$0 = 2 - 2$

the more common 2nd order tensor product is

4b

$(TS)_{ij} = \underbrace{T_{ik} \; S_{kj}}$

only 1 index is contracted

- As we can see for a tensor order m multiplying a tensor order n (n <= m), we can contract n indices (common definition), n - 1 indices, …, 1 index
- For S & T both second order (n = m = 2) the common tensor product only contracts 1 index (4b)

Identity tensor in higher dimensions

$V = \overset{(2)}{I} \; V$

$V \downarrow$ vector, $V \downarrow$ vector

$1 = \overset{?}{2} - 1$

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\overset{m}{T} = \overset{2m}{I} \; \overset{m}{T}$

$\overset{m}{T} \downarrow$ m order tensor

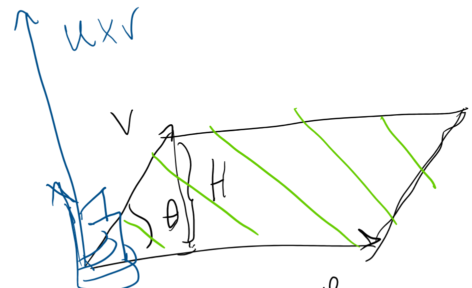
$m = \left(\overset{?}{2m} \right) - m$

For example we have $(\overset{4}{I})$ $\quad \overset{4}{I}_{ijkl} = \ldots$

1.13 Vector/cross/exterior product

$u \times v =$ Vector

$|u \times v| = |u||v| \sin\theta$ = dashed area



$H = |v| \sin\theta$

Area $= H|u| = |u||v|\sin\theta$

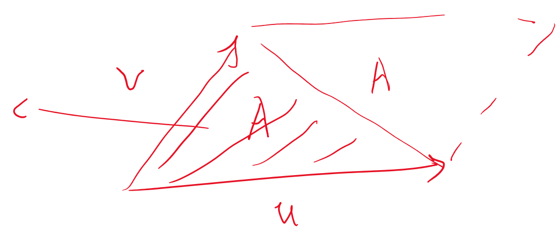
recall

scalar value $u \cdot v = |u||v|\cos\theta = [u_1 \; u_2 \; u_3]\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ Scalar

$u \otimes v = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} [v_1 \; v_2 \; v_3]$ 2nd order tensor

Area of triangle

$\frac{1}{2} |u \times v|$



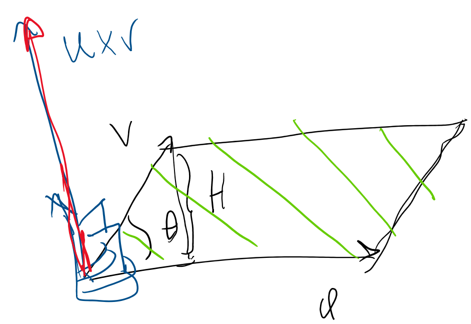
$2A = |u \times v|$

$u \times v$: magnitude = green area

direction : normal to u & v

$\Rightarrow$ normal to u & v plane

$u \times v$ = surface normal vector for quad formed by u & v



④

Formula for $u \times v$

Recall: $E = \epsilon_{ijk} \, e_i \otimes e_j \otimes e_k$ alternating tensor

$u \times v = (E v) u$

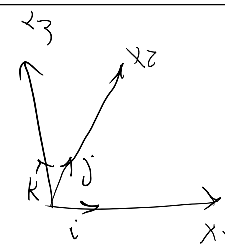
$$\underbrace{u \times v}_{\text{vector}} = (\underset{\substack{\text{3rd}\\\text{order}}}{E}\,\underset{\substack{\text{1st}\\\text{order}}}{v})\,\underset{\text{1st order}}{u}$$

$$3 - 1 = 2$$

$$1 = 2 - 1$$

We can show

$$u \times v = \det \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2 v_3 - u_3 v_2)\, i + (v_1 u_3 - u_1 v_3)\, j + (u_1 v_2 - v_1 u_2)\, k$$



$i = e_1$
$j = e_2$
$k = e_3$

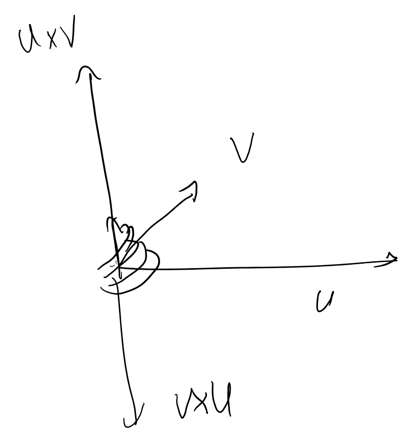
$$u \times v = (E v) u = \epsilon_{ijk}\, u_i\, v_j\, e_k$$

5

Interesting facts

$$v \times u = -u \times v$$

$$u \times (v \times w) \neq (u \times v) \times w$$



Theorem 93 *The vector product is* not associative*:*

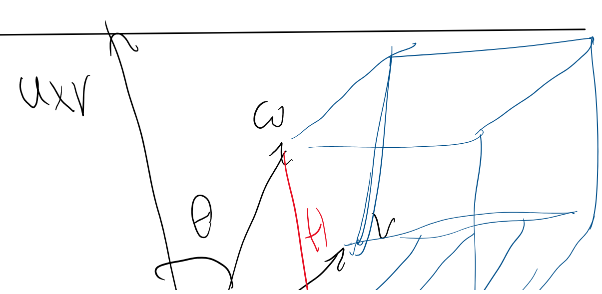
$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \neq \mathbf{u} \times (\mathbf{v} \times \mathbf{w}),$$

indeed

$$\begin{aligned} (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} &= (\mathbf{u} \cdot \mathbf{w})\, \mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\, \mathbf{u}, \\ \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= (\mathbf{u} \cdot \mathbf{w})\, \mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\, \mathbf{w}. \end{aligned}$$

Triple product: Used to calculate volumes

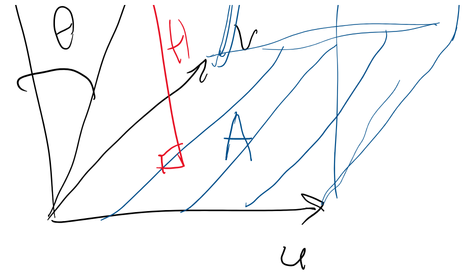
$$\bar{V} = (u \times v) \cdot w$$



$$V = (u \times v) \cdot w$$

volume

$$= \underbrace{|u \times v|}_{A} \, |w| \cos\theta$$

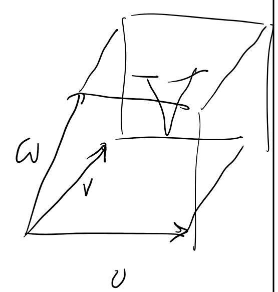


$$V = AH$$

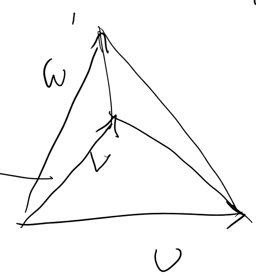
$$= A|w| \cos\theta$$

$$V = (u \times v) \cdot w = \begin{vmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \cdot (w_1 e_1 + w_2 e_2 + w_3 e_3) \Longrightarrow$$

$$V = (u \times v) \cdot w = \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$



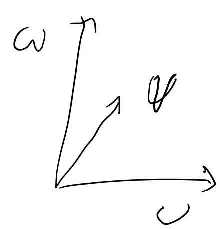
$$V_{tet} = \frac{(u \times v) \cdot w}{2 \times 3} = \frac{(u \times v) \cdot w}{6}$$



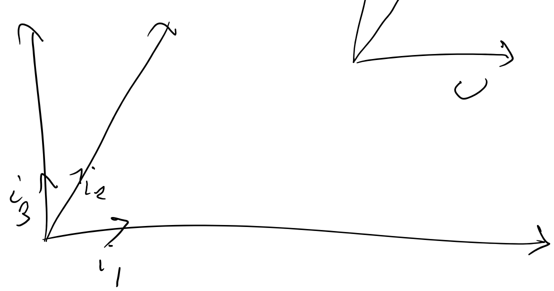
6

u, v, w are called right-hand oriented iff

$$(u \times v) \cdot w > 0$$



$$(i_1 \times i_2) \cdot i_3 = 1$$



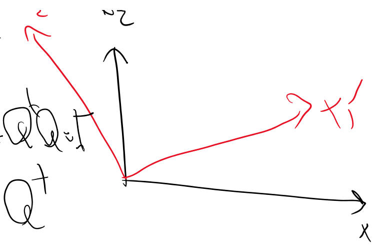
We generally use Right-handed coordinate systems

Orthonormal or orthogonal tensors:

recall Q matrix x_2' x_2

Orthonormal or orthogonal tensors:

recall the matrix



$$QQ^t = Q^tQ = I$$
$$Q^{-1} = Q^t$$

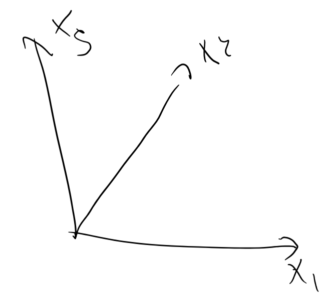
A second order tensor is called orthonormal if its inverse is equal to its transpose

$$TT^t = T^tT = I \qquad (7)$$
$$T^t = T^{-1}$$

What do orthogonal tensors represent?

$$[T] = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{ccc} c_1 & c_2 & c_3 \\ \end{array}$$

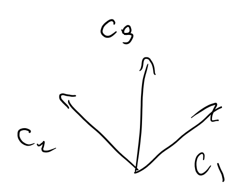
$$[T] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array}$$



$$T^tT = \begin{bmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} =$$

$$= \begin{bmatrix} c_1\cdot c_1 & c_1\cdot c_2 & c_1\cdot c_3 \\ c_2\cdot c_1 & c_2\cdot c_2 & c_2\cdot c_3 \\ c_3\cdot c_1 & c_3\cdot c_2 & c_3\cdot c_3 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boxed{c_i \cdot c_j = \delta_{ij}} \qquad (8a)$$



forms an orthonormal coordinate system

$$TT^t = I \Longrightarrow \boxed{r_i \cdot r_j = \delta_{ij}} \qquad (8b)$$

8: Rows of an orthonormal matrix are mutually normal to each other and each with magnitude 1.

Columns of an orthonormal matrix are mutually normal to each other and each with magnitude 1.