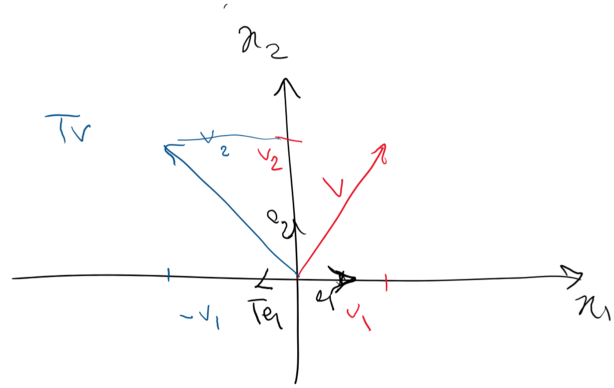
CM2023/10/02

Monday, October 2, 2023 9:46 AM

Orthogonal tensors represent a rotation plus possibly a reflection.

Example of a reflection:

Tv rotation of v w.r.t x_2 axis



$$T = \underbrace{\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}}$$

$$\underbrace{\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{e_1} = T e_1 = \begin{bmatrix} T_{11} \\ T_{21} \end{bmatrix}$$

To form tensor T in coordinate (x_1, x_2) act T on unit vector e_i and this will be column i

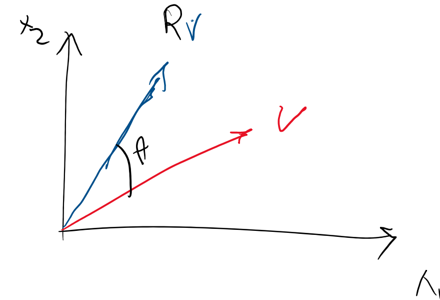
$$T e_1 = \underbrace{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{c_1} \qquad T e_2 = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{c_2} \qquad [T] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\det T = (-1)(1) = -1$$

$$\boxed{\begin{array}{l} T \text{ orthogonal} \\ \det T = -1 \iff \text{having 1 reflection} \end{array}}$$

Example of a rotation:

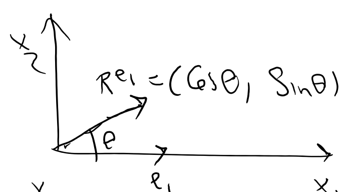
R represents rotation by θ in counter clockwise direction



Components of R in (x_1, x_2) coordinate system

column 1 of R

$$c_1 = R e_1 = \begin{bmatrix} c \\ s \end{bmatrix}$$

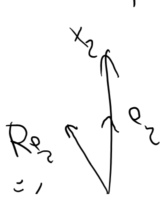


$Re_1 = (\cos\theta, \sin\theta) = (c, s)$

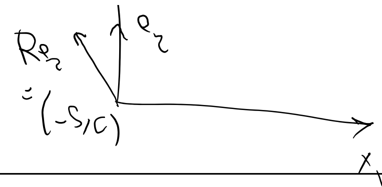
$c := \cos(\theta)$

$s := \sin(\theta)$

$$c_2 = R e_2 = \begin{bmatrix} -s \\ c \end{bmatrix}$$



$$c_2 = R e_2 = \begin{bmatrix} -s \\ c \end{bmatrix}$$



$$R = \left[c_1 \,|\, c_2 \right] \Rightarrow$$

$$\boxed{R = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}}$$

rotation in 2D by θ in counterclockwise direction

$\det R = c^2 + s^2 = 1$

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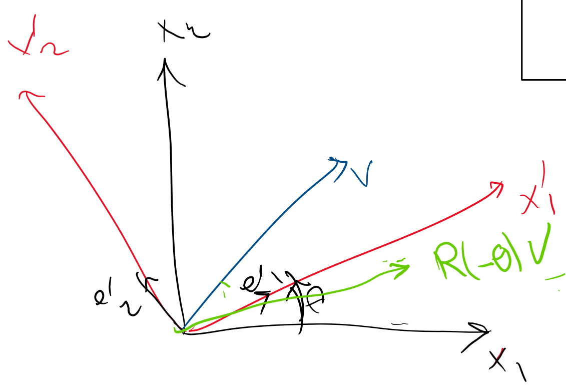

$$[v'] = Q[v] \qquad Q = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

$$[R(-\theta)v] = [v']$$

$$S = RT \qquad \det S = \underbrace{\det R}_{1}\ \underbrace{\det T}_{-1} = -1$$

$S \downarrow$ orthogonal

$$SS^t = RT(T^tR^t) = R(\underbrace{TT^t}_{I})R^t = RR^t = I$$

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Any orthogonal 2nd order tensor S represents a rotation (angle of rotation can be zero) & possibly a reflection

$\det S = -1 \iff$ has 1 reflection

"Large deformation" rotations are represented by orthogonal tensors

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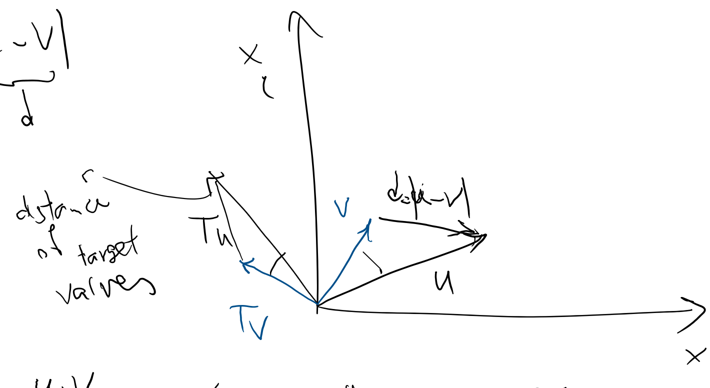
Other properties of orthogonal 2nd order tensors

1. $T \in \text{Orth}\,\mathcal{V}$ (T is orthogonal)

2. $\forall u, v \quad Tu \cdot Tv = u \cdot v$ "$\cdot$" (inner product) is preserved

3. $\forall u \quad |Tu| = |u|$ magnitude is preserved

4. $\forall u, v \quad |Tu - Tv| = |\underbrace{u - v}_{d}|$



$1 \iff 2$

$Tu \cdot Tv = \underbrace{T^t T}_{} u \cdot v = u \cdot v$ (recall $\omega \cdot Tv = T^t \omega \cdot v$) $\omega = Tu$

T orthonormal

$T^t T = I$

$2 \Rightarrow 3$ $\forall u, v$ ② $Tu \cdot Tv = u \cdot v$ choose $v = u$ $\Bigg\} \Rightarrow$ $Tu \cdot Tu = u \cdot u \Rightarrow |Tu|^2 = |u|^2 \Rightarrow$

$|Tu| = |u|$

$3 \Rightarrow 2$

$|Tu| = |u|$

$|Tv| = |v|$

$|T(u+v)| = |u+v|$

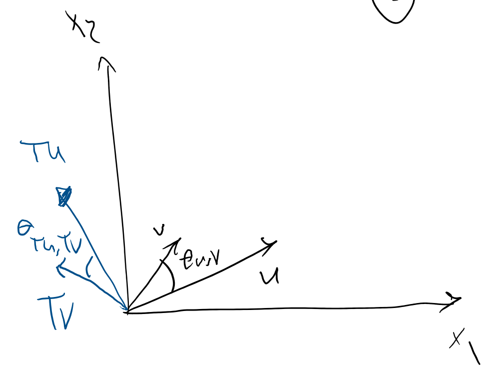
Square all of this & add/subtract them $\Bigg\} \Rightarrow Tu \cdot Tv = u \cdot v$

$3 \Rightarrow 4$ $\forall \omega$ ③ $|T\omega| = |\omega|$ choose $\omega = u - v$ $\Rightarrow |T(u-v)| = |u-v| \Rightarrow$

$|Tu - Tv| = |u - v|$

$4 \Rightarrow 3$ $\quad \forall\, u,v \quad |Tu - Tv| = |u-v| \quad \Rightarrow \quad |Tu| = |u|$ (3)

choose $v = 0$



Orthogonal tensors also preserve angle between two vectors:

$$\cos\theta_{Tu,Tv} = \frac{Tu\cdot Tv}{|Tu|\,|Tv|} \overset{\text{Prop (2)}}{=} \frac{u\cdot v}{|u|\,|v|} = \cos\theta_{u,v}$$

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$$\theta_{Tu,Tv} = \theta_{u,v}$$

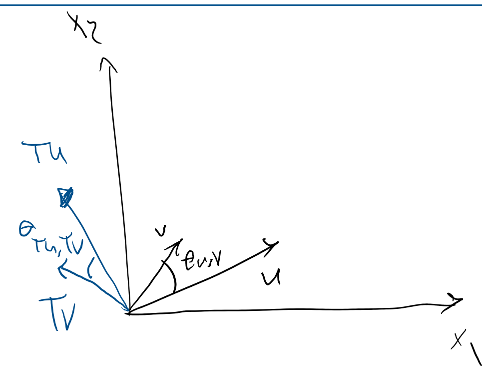
(4)

Summary:

Orthogonal tensors

- Preserve the magnitude of vectors $\quad |Tu| = |u|$
- Preserve:
  - Angle $\quad \theta_{Tu,Tv} = \theta_{u,v}$
  - Distance $\quad |Tu - Tv| = |u-v|$
  - Inner product $\quad Tu\cdot Tv = u\cdot v$

  Between any two vectors



Skew-symmetric tensors

They also represent rotations (but small rotations)

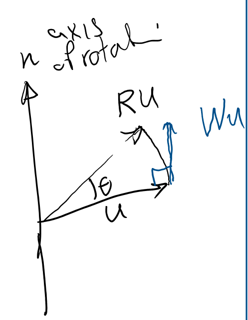
$$W = -W^t$$

$$u\cdot Wu = \underbrace{W^t}_{-W} u\cdot u = -Wu\cdot u = -u\cdot Wu$$

$\Rightarrow$

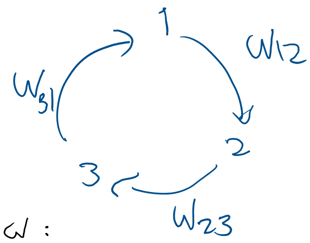
$u\cdot Wu = 0$ for skew sym W ($W^t = -W$)

(5) $\quad u \perp Wu$



Definition of axis of a second order skew-symmetric tensor:

$$W = \begin{bmatrix} 0 & W_{12} & -W_{31} \\ -W_{12} & 0 & W_{23} \\ W_{31} & -W_{23} & 0 \end{bmatrix}$$



we have 3 indepent values of w:

(W_{12}, W_{23}, W_{31})

$$Wu = \underline{\begin{bmatrix} 0 & W_{12} & -W_{31} \\ -W_{12} & 0 & W_{23} \\ W_{31} & -W_{23} & 0 \end{bmatrix}} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} W_{12} u_2 - W_{31} u_3 \\ -W_{12} u_1 + W_{23} u_3 \\ W_{31} u_1 - W_{23} u_2 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} W_{23} \\ W_{31} \\ W_{12} \end{bmatrix}}_{\omega = ax(W)} \times \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\boxed{\begin{aligned} &\omega_i = -\tfrac{1}{2} \epsilon_{ijk} W_{jk} && \omega = ax(W) = \begin{bmatrix} W_{23} \\ W_{31} \\ W_{12} \end{bmatrix} \\ &W_{ij} = -\epsilon_{ijk} \omega_k && W = ax(\omega) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} \\ &Wu = \omega \times u \end{aligned}}$$

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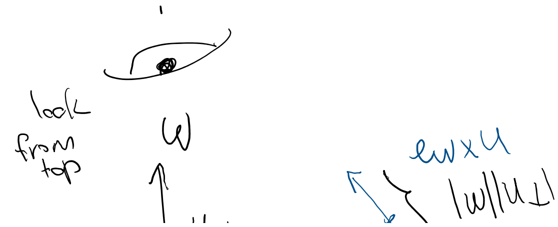
Why $Wu = \omega \times u$ is a small angle rotation?

$$\omega \times u = |\omega| \underbrace{(e_\omega \times u)}$$



$$\omega \times u = |\omega|(\underbrace{e_\omega \times u})$$

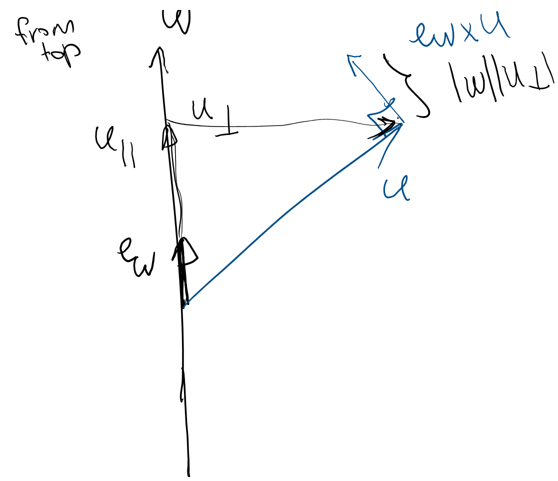
$\omega = |\omega| e_\omega$

$u = u_{\parallel} + u_{\perp}$

$u_{\parallel} \parallel e_\omega$: $u_{\parallel} = (u \cdot e_\omega) e_\omega$

$u_{\perp} \perp e_\omega$ $u_{\perp} = u - u_{\parallel}$

$|e_\omega \times u| = |u_{\perp}||\omega|$



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$$|\omega \times u| = \underbrace{|\omega|}_{\theta} \underbrace{|u_\perp|}_{\ell} = \ell\theta$$

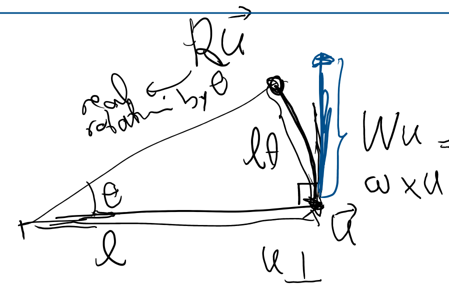
$|\omega|$ = angle of rotation

$|u_\perp|$ = distance to angle of rotation

$e_\omega = \frac{\omega}{|\omega|}$ axis of rotation

$$\underbrace{\underbrace{Ru}_{\text{exact rotation by } \theta} - \underbrace{u}_{\text{initial position}}}_{\text{arc}} \approx \omega \times u$$

small rotation
change of position



Decomposition of a tensor to symmetric and skew-symmetric tensors

$T = \text{Sym}(T) + \text{skew}(T)$

we'll see that for small def gradient

$\text{sym}(T) = \frac{T + T^t}{2}$ $\longrightarrow$ strain $\left(E = \frac{\nabla u + \nabla u^T}{2}\right)$

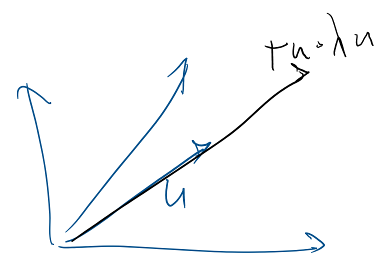
$$\text{skew}(T) = \frac{T - T^t}{2} \qquad \text{rotation } \left(W = \frac{\nabla u - \nabla u^t}{2}\right)$$

We deal with several symmetric tensors (strain E, stress σ)

We want to calculate the eigenvalues of symmetric tensors (e.g. principal strains, stresses)

Recall definitions of eigenvalues and eigenvectors

$$\underbrace{T}_{\text{2nd order tensor}}\ \underbrace{u}_{\text{eigenvector}} = \lambda u \qquad \lambda \in \mathbb{C} \text{ (Complex \#'s) eigenvalue}$$



Look at notes from 8/28

column i is eigen vector i

② $\boxed{AU = U\Lambda} \Longrightarrow$ diagonal i is eigen vector i

multiply by U^{-1} on the right

$AUU^{-1} = U\Lambda U^{-1} \Longrightarrow$ ③ $\boxed{A = U\Lambda U^{-1}}$

diagonalizing a matrix

$$U = \begin{bmatrix} u^{(1)} & u^{(2)} & u^{(3)} \end{bmatrix}$$

eigenvector 1 2 3

$$\boxed{f(A) = U \begin{pmatrix} f(\lambda^{(1)}) & & \\ & f(\lambda^{(2)}) & \\ & & \ddots & \\ & & & f(\lambda^{(n)}) \end{pmatrix} U^{-1}}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$$

Not all matrices can be diagonalized

- If the matrix has distinct eigenvalues -> It is diagonalizable
- If some eigenvalues are repeated (e.g. lamba_1 = lambda_2 = 5) it depends (Jordan form ...).
- If the matrix is symmetric (or more general Hermitian for complex matrices) it is not only diagonalizable but 1) eigenvalues are real, 2) eigenvectors are normal to each other.