CM2023/10/04

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Properties of eigenvectors and eigenvalues of symmetric 2nd order tensors (or matrices)

1. Eigenvalues are real:

$$\underset{\text{eigenvector}}{A u} = \underset{\text{eigenvalue}}{\lambda} u \quad (1)$$

$\overline{(\ )}$ conjugate

$z = a + bi \longrightarrow \bar{z} = a - bi$

$|z|^2 = z\bar{z} = a^2 + b^2$

(1) $\bar{u} \cdot A u = \lambda \bar{u} u = |u|^2$

transpose

$u \cdot A^T \bar{u} = \lambda |u|^2$

conjugate

$\bar{u} \bar{A}^T u = \bar{\lambda} |u|^2$

conjugate transpose

$$\boxed{\bar{A}^t = A}$$

Hermitian

$(A^t = A)$ for real sym matrix

$\bar{u} A u = \lambda |u|^2 = \bar{\lambda} |u|^2$

$\Rightarrow \lambda = \bar{\lambda} \Rightarrow \boxed{\lambda \text{ Real}}$

FYI: these properties hold for a more general class of Hermitian matrices
https://en.wikipedia.org/wiki/Hermitian_matrix

$$\boxed{A \text{ Hermitian} \iff a_{ij} = \overline{a_{ji}}}$$

or in matrix form:

$$A \text{ Hermitian} \iff A = \overline{A^{\mathsf{T}}}.$$

Hermitian matrices can be understood as the complex extension of real symmetric matrices.

If the conjugate transpose of a matrix A is denoted by A^{H}, then the Hermitian property can be written concisely as

$$\boxed{A \text{ Hermitian} \iff A = A^{\mathsf{H}}}$$

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2. For an n by n symmetric matrix we have n linearly independent eigenvectors and in fact they are mutually normal to each other.

Given $A = A^t$

we have $\lambda_1 \neq \lambda_2$ want to prove corresponding eigenvectors are normal

Given $A = A$

we have $\lambda_1 \neq \lambda_2$ & want to prove corresponding eigenvectors are normal

$$\left.\begin{array}{ll} i) & A u_{(1)} = \lambda_1 u_{(1)} \\ ii) & A u_{(2)} = \lambda_2 u_{(2)} \end{array}\right\} \Longrightarrow \quad u_{(1)} \perp u_{(2)}$$

i) $u_{(2)} \cdot A u_{(1)} = \lambda_1 \, u_{(2)} \cdot u_{(1)}$

ii) $\underbrace{u_{(1)} \cdot A u_{(2)}}_{\parallel} = \lambda_2 \, u_{(1)} \cdot u_{(2)}$

$\underbrace{A^t}_{\parallel} u_{(1)} \cdot u_{(2)}$

$A \quad (A = A^t)$

$$\longrightarrow \left.\begin{array}{l} u_{(2)} \cdot A u_{(1)} = \lambda_1 u_{(1)} \cdot u_{(2)} \\ u_{(2)} \cdot A u_{(1)} = \lambda_2 u_{(1)} \cdot u_{(2)} \end{array}\right\}$$

subtract

$$\Longrightarrow \left.\begin{array}{l} (\lambda_1 - \lambda_2)\, u_{(1)} \cdot u_{(2)} = 0 \\ \lambda_1 \neq \lambda_2 \end{array}\right\} \Longrightarrow$$

$$\boxed{u_{(1)} \cdot u_{(2)} = 0 \qquad u_{(1)} \perp u_{(2)}}$$

In fact, we can make eigenvectors orthonormal (their magnitude is one and they are normal to each other)

$A_{3\times 3}$

$$(*) \quad \begin{array}{l} A u_{(1)} = \lambda_1 u_{(1)} \\ A u_{(2)} = \lambda_2 u_{(2)} \\ A u_{(3)} = \lambda_3 u_{(3)} \end{array}$$

$e_1^* = \dfrac{u_1}{|u_1|} \longrightarrow A e_1^* = \lambda_1 e_1^*$

$e_2^* = \dfrac{u_2}{|u_2|}$

$e_3^* = \dfrac{u_3}{|u_3|}$

why?

$A e_1^* = A \dfrac{u_1}{|u_1|} = \dfrac{1}{|u_1|} A u_1 \underset{(*)}{=}$

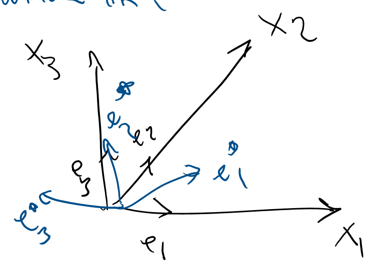
$\dfrac{1}{|u_1|} \lambda_1 u_1 = \lambda_1 \dfrac{u_1}{|u_1|} = \lambda_1 e_1^*$

(†)

$e_i^* = \dfrac{u_{(i)}}{|u_{(i)}|}$ no summation on i

$A e_i^* = \lambda_i e_i^*$

$e_i^* \cdot e_j^* = 0$



2D and 3D examples, also discussing what happens when eigenvalues are equal:

2D

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

$$Au = \lambda u = \lambda I u \Rightarrow$$

$$\left.\begin{array}{l}(A - \lambda I)u = 0 \\ u \neq 0\end{array}\right\} \Rightarrow \det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} a-\lambda & b \\ b & d-\lambda \end{bmatrix} = 0 \Rightarrow (a-\lambda)(d-\lambda) - b^2 = 0$$

$$\lambda^2 - \underbrace{(a+d)}_{\text{trace} A}\lambda + \underbrace{ad - b^2}_{\det A} = 0$$

$$\lambda = \frac{(a+d) \pm \sqrt{(a-d)^2 + 4b^2}}{2}$$

$$\lambda_1 = \lambda_2 \iff \sqrt{\underbrace{(a-d)^2}_{\geq 0} + \underbrace{4b^2}_{\geq 0}} = 0 \iff \begin{array}{l} a = d \\ b = 0 \end{array}$$

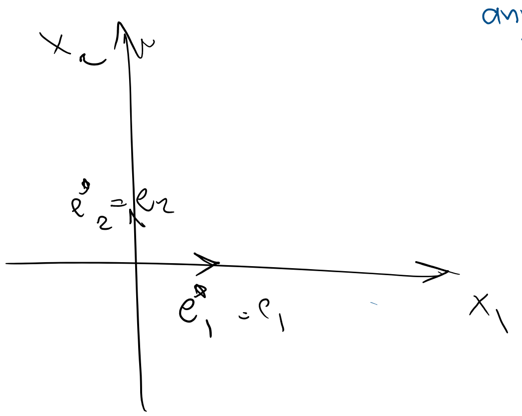
$$\iff A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \qquad \lambda_{1,2} = a$$

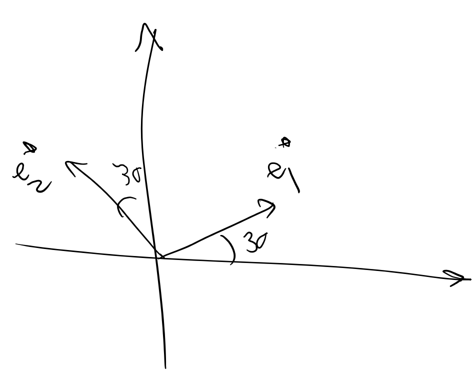
What are the eigenvectors of this matrix?

$A = aI$

$Au = aIu = au$

any vector is an eigenvector with eigenvalue a





3D:

3D:

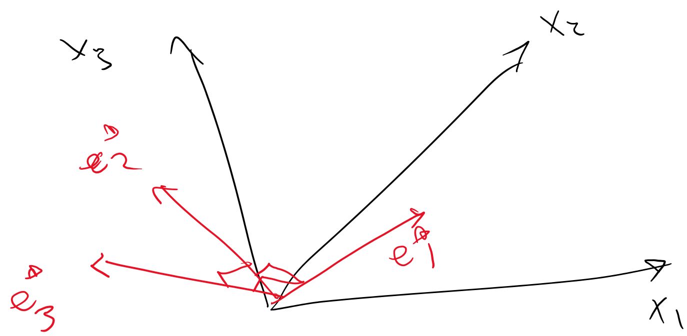
$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \longrightarrow \lambda_1, \lambda_2, \lambda_3 \quad \text{solved from}$$

$\det(A - \lambda I) = 0$

3rd order equation

we have 3 cases:

1) All eigenvalues are distinct $\quad \lambda_1 \neq \lambda_2 \quad \lambda_2 \neq \lambda_3 \quad \lambda_1 \neq \lambda_3$



$A\vec{e}_i = \lambda_i \vec{e}_i$ no summation on i

$\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$

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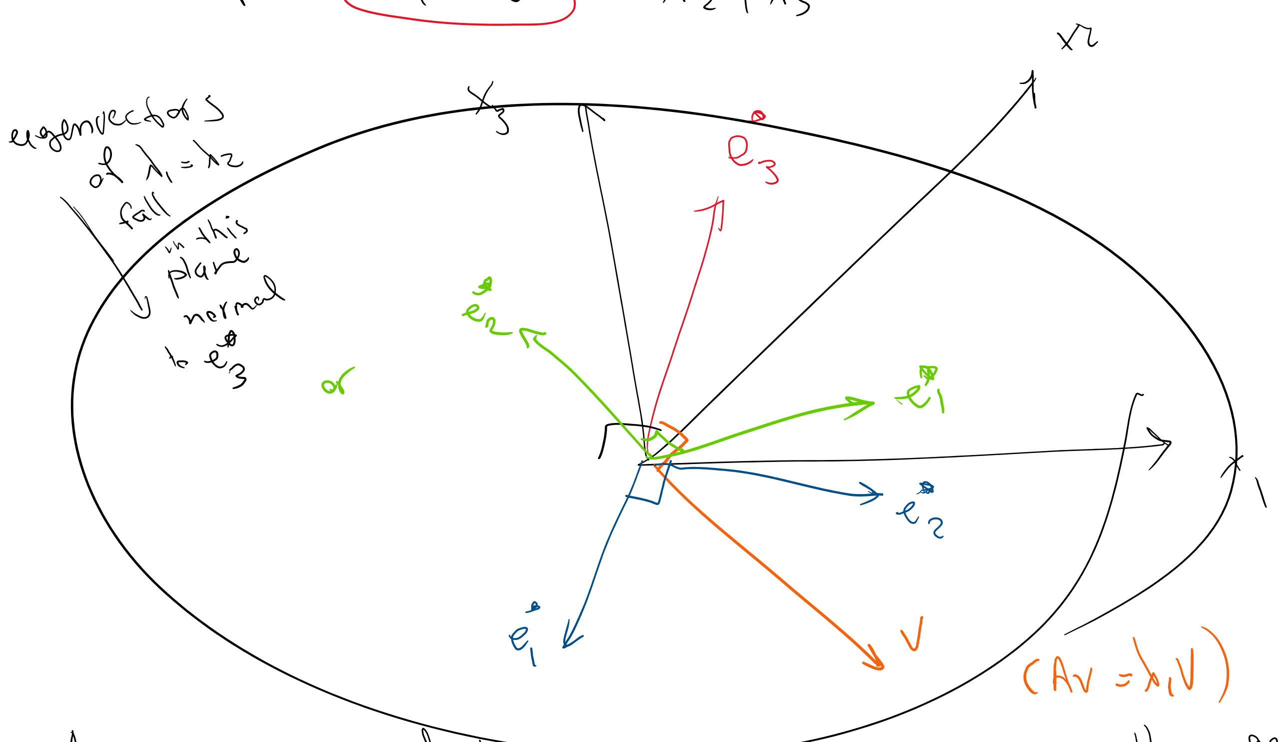
2) Two of the eigenvalues are equal

For example $\underbrace{\lambda_1 = \lambda_2}$ $\quad \lambda_2 \neq \lambda_3$



Any vector normal to $\vec{e}_3$ is an eigenvector with eigenvalue λ_1

Any vector normal to e^*_3 is an eigenvector with eigenvalue λ_1

We can choose any two mutually unit normal vectors in the plane normal to e*3 to form 3 mutually normal unit vectors (a coordinate system)

(e^*_1, e^*_2, e_3) (e^*_1, e^*_2, e^*_3), ...

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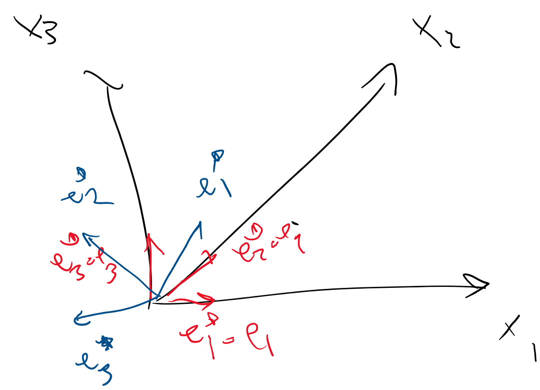
3) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$

$$A = U^{-1} \Lambda U = U^{-1} \begin{bmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{bmatrix} U = \lambda U^{-1} I U = \lambda I$$

$$A = \begin{bmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{bmatrix} = \lambda \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

$Au = \lambda u$
for ANY u



(2)
Summary:
For a symmetric matrix A we can find 3 mutually normal unit vectors (a coordinate system) from eigenvectors of A.
If eigenvalues are distinct => (e^*_1, e^*_2, e^*_3) is unique
Otherwise, we have many choices

Formula for calculating eigenvalues of a symmetric matrix in 3D:

$S_{3\times3}$

$$(S - \underset{\text{eigenvalue}}{\sigma} I)\underset{\text{eigenvector}}{u} = 0 \qquad \det(S - \sigma I) = 0$$

$$\det \begin{bmatrix} S_{11}-\sigma & S_{12} & S_{13} \\ S_{12} & S_{22}-\sigma & S_{23} \\ S_{13} & S_{23} & S_{33}-\sigma \end{bmatrix} = 0$$

Cayley-Hamilton equation or the characteristic equation for the 3x3 symmetric matrix

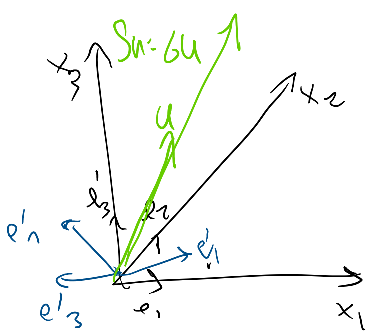
(CH)

$$-\sigma^3 + I_1\sigma^2 - \underline{I_2}\sigma + I_3 = 0$$

$$I_1 = \text{trace}(S) = S_{11} + S_{22} + S_{33} = S_{ii}$$

$$I_2 = \frac{1}{2}\left[(\text{trace}(S))^2 - \text{trace}(S^2)\right] = \frac{1}{2}(S_{ii}S_{jj} - S_{ij}S_{ij})$$

$$I_3 = \det S$$



$[S]$ e_1, e_2, e_3

$[S']$ " e'_1, e'_2, e'_3 $\qquad S'_{ij} = Q_{im}Q_{jn}S_{mn}$

(3)

- Eigenvalues and eigenvectors of a tensor DO NOT depend on the choice of coordinate system (note: eigenvector is a vector and its components transform as a vector)
- I1, I2, I3 are called the invariants of a symmetric 3x3 tensor.

There are many representation theorem in continuum mechanics:

$$\overset{\text{internal energy}}{f}(\underset{\text{stress}}{S}) = f(S_{11}, S_{12}, S_{13}, S_{22}, S_{23}, S_{33})$$

stress

scalar $= f_\sigma(\sigma_1, \sigma_2, \sigma_3)$

OR $= f_I(I_1, I_2, I_3)$

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ & S_{22} & S_{23} \\ \text{sym} & & S_{33} \end{bmatrix}$$

Instead of calibrating the equation for 6 arguments we can calibrate it for 3 arguments:

- Either eigenvalues (sigma1, sigma2, sigma3)
- Or principal values (I1, I2, I3)

Note:
FYI:
The actual matrix satisfies CH equation

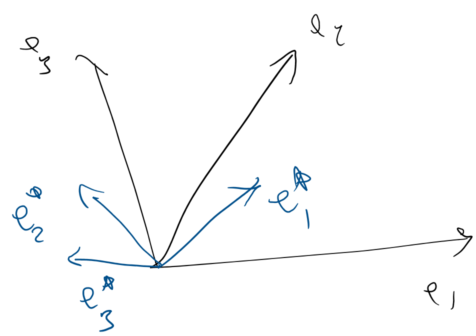
$$-S^3 + I_1 S^2 + I_2 S + I_3 I = 0$$

Hint $S = U^{-1} \Lambda U$ plug in CH

Representation of a symmetric 2nd order tensor in it's principal direction coordinate:

S sym we have found

3 orthonormal eigenvectors



$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ & S_{22} & S_{23} \\ \text{sym} & & S_{33} \end{bmatrix}_{e_1, e_2, e_3}$$

e_1^*, e_2^*, e_3^* eigenvectors also called principal direction)

$S e_i^* = \lambda_i e_i^*$ no summation on i

$$[S^*] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}_{e_1^*, e_2^*, e_3^*}$$

why recall $T_{ij} = e_i \cdot T e_j$

$$S^*_{ij} = e^*_i \cdot (\underbrace{S e^*_j}) = e^*_i \cdot (\lambda_j e^*_j) = \lambda_j \, e^*_i \cdot e^*_j = \lambda_j \delta_{ij}$$

no summation on j $\longrightarrow$ " $\longrightarrow$ "

$$[S^*] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

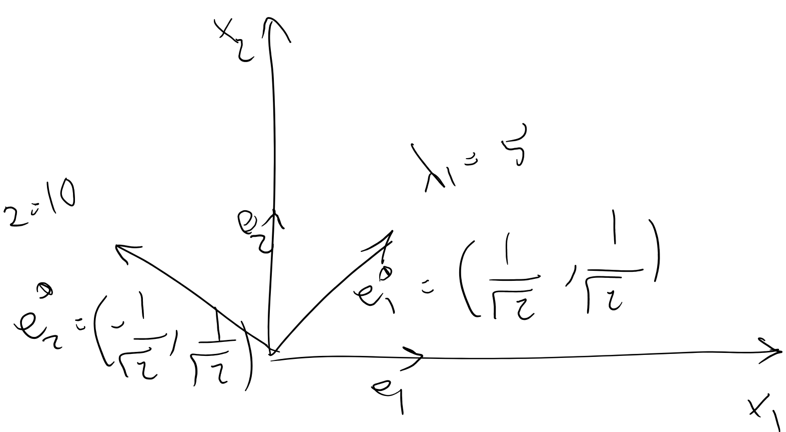
$$\boxed{[S^*] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \qquad S = S^*_{ij} \, e^*_i \otimes e^*_j = \sum_{i=1}^{3} \lambda_i \, e^*_i \otimes e^*_i}$$

(4)

Example:

$$[S^*] = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$$

S in e^*_1, e^*_2 coordinate



$$= 5 \, e^*_1 \otimes e^*_1 + 10 \, e^*_2 \otimes e^*_2$$

$[S]$ $\rightarrow$ components of S in e_1, e_2 system