CM2023/10/11

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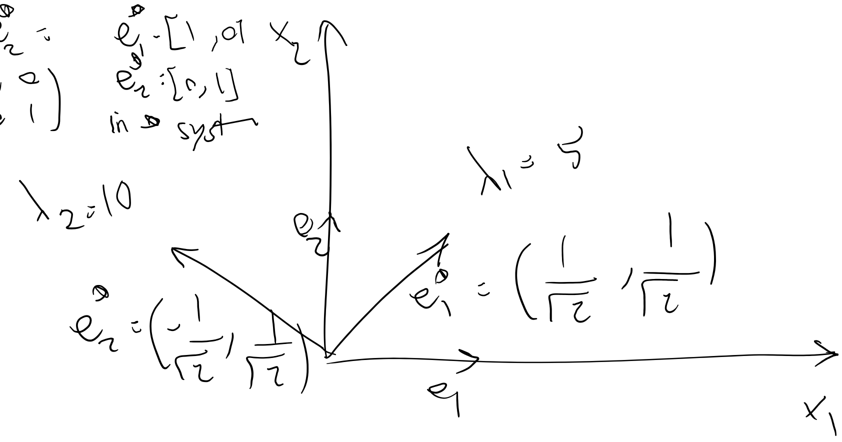
Example: $\lambda_1 e_1^* \otimes e_1^* + \lambda_2 e_2^* \otimes e_2^* =$ $5\begin{bmatrix}1&0\\0&0\end{bmatrix} + 10\begin{bmatrix}0&0\\0&1\end{bmatrix}$

$e_1^* = [1, 0]$, $e_2^* = [0, 1]$ in * syst



$$[S^*] = \begin{bmatrix}5 & 0\\ 0 & 10\end{bmatrix}$$

S in e_1^*, e_2^* coordinate

$$= 5\, e_1^* \otimes e_2^* + 10\, e_2^* \otimes e_2^*$$

$[S]$ → components of S in e_1, e_2 system

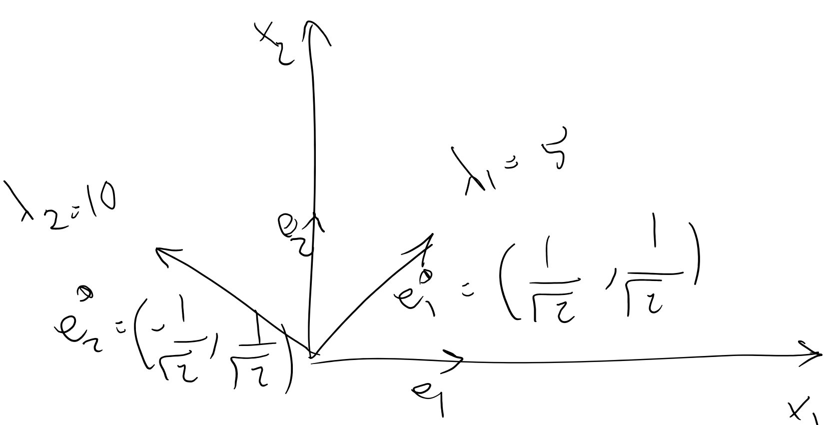
One way is to form Q matrix and do the coordinate transformation rule

$$[S^*] = Q[S]Q^t \iff [S] = Q^t[S^*]Q$$

we can use this

or easier way here:

$$S = \sum_{i=1}^{2} \lambda_i e_i \otimes e_i = \lambda_1 e_1 \otimes e_1 + \lambda_2 e_2 \otimes e_2$$



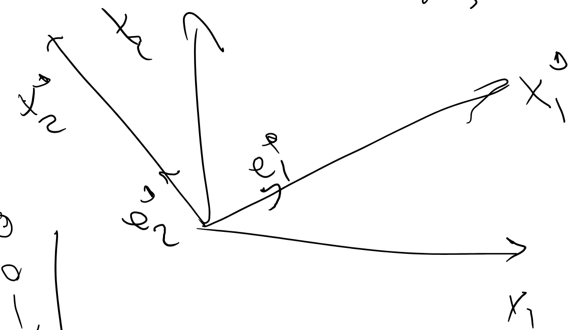
$$= 5\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$$

$$+ 10\begin{bmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$$

$$= 5\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix} + 10\begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{bmatrix} = \begin{bmatrix}\frac{15}{2} & -\frac{5}{2}\\ -\frac{5}{2} & \frac{15}{2}\end{bmatrix}$$

Example 2

$$[\mathbf{S}] = \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \sigma_3 \end{bmatrix} \qquad \det S = \sigma_1 \sigma_2 \sigma_3$$



$$S = \sum \sigma_i \, \vec{e}_i \otimes \vec{e}_i$$

$$S^{-1} = \sum_{i=1}^{3} \frac{1}{\sigma_i} \vec{e}_i \otimes \vec{e}_i \qquad [S^{-1}]^{\mathbf{e}} = \begin{bmatrix} 1/\sigma_1 & 0 & 0 \\ 0 & 1/\sigma_2 & 0 \\ 0 & 0 & 1/\sigma_3 \end{bmatrix}$$

$$\operatorname{Ln} S = \sum_{i=1}^{3} \operatorname{Ln} \sigma_i \, \vec{e}_i \otimes \vec{e}_i \qquad [\operatorname{Ln} S]^{\mathbf{e}} = \begin{bmatrix} \operatorname{Ln} \sigma_1 & & \\ & \operatorname{Ln} \sigma_2 & \\ & & \operatorname{Ln} \sigma_3 \end{bmatrix}$$

Positive definite tensors:
Background discussion

$$u \cdot Tu = u \cdot (\operatorname{sym} T + \operatorname{skew} T) u \qquad \operatorname{sym} T = \frac{T + T^t}{2}$$

$$= u \cdot \operatorname{sym} T u + u \cdot \operatorname{skew} T u \qquad \operatorname{skew} T = \frac{T - T^t}{2}$$

$$= \underbrace{u_i (\operatorname{sym} T)_{ij} u_j}_{} + \underbrace{u_i (\operatorname{skew} T)_{ij} u_j}_{}$$

$$\underbrace{\underbrace{u_i u_j}_{\text{sym in } ij} \quad \underbrace{(\operatorname{skew} T)_{ij}}_{\text{skew sym}}}_{0} \qquad (\operatorname{skew} T)_{ij} = -(\operatorname{skew} T)_{ji}$$

(1)
$$\boxed{\begin{array}{l} u \cdot Tu = u \cdot (\operatorname{sym} T) u \\ \text{only the symmetric part of } T \text{ contributes to } u \cdot Tu \end{array}}$$

Skew part of T does not contribute to u.Tu

Positive or positive-definite tensor:

$$\underbrace{\forall u \quad u \cdot Tu \geq 0}_{\text{positive}} \quad \& \quad \underbrace{u \cdot Tu = 0 \iff u = 0}_{\substack{\text{if in addition this condition holds} \\ \text{it's called positive definite}}}$$

Because of (1) we only need to check positive-definiteness for the symmetric part of T
Examples:

For symmetric matrices we can diagonalize them and express them in their eigen-value (principal) direction.

$$[T]^{*} = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \qquad \lambda_1 = 5, \quad \lambda_2 = 0$$

$$(u)^{*} \cdot T^{*} u^{*} = \begin{bmatrix} u_1^{*} & u_2^{*} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} u_1^{*} \\ u_2^{*} \end{bmatrix} = \lambda_1 u_1^{*2} + \lambda_2 u_2^{*2} \quad (2)$$

here $\lambda_1 = 5$ $\lambda_2 = 0$ so

(2) $\Rightarrow$ $u \cdot Tu = 5 u_1^{*2} \geq 0$ "positive"

How about $[u]^{*} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $u \cdot Tu = 0$ but $[u]^{*} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq 0$

positive but not pos. def

$$T = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \qquad \lambda_1 = 5, \quad \lambda_2 = 3$$

$$u \cdot Tu = 5(u_1^{*})^2 + 3(u_2^{*})^2 \geq 0$$

$$\& \quad u \cdot Tu = 0 \iff [u]^{*} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

pos. def.

$T = \begin{bmatrix} 5 & 0 \\ \end{bmatrix}$ $\quad u \cdot Tu = 5(u_1^{*})^2 - 3(u_2^{*})^2$

$$T = \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix} \qquad u \cdot Tu = 5(u_1^*)^2 - 3(u_2^*)^2$$

$$[u]^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad u \cdot Tu = -3 < 0 \qquad \text{not positive}$$

(2)

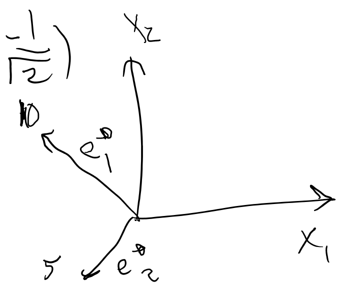
A 2nd order tensor is positive if all its eigenvalues are greater than equal to zero $\lambda_i \geq 0$

It's positive definite-definite if .... Are greater than zero $\lambda_i > 0$

$$T = \begin{bmatrix} 15/2 & -5/2 \\ -5/2 & \frac{15}{2} \end{bmatrix}$$ calculate eigen values & vectors

$t_1 = 10$, $t_2 = 5$

$e_1^* = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $e_2^* = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$



$$[T]^* = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\longrightarrow [\sqrt{T}]^* = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{5} \end{bmatrix}, \sqrt{T} \text{ in } e_1 \& e_2 =$$

$$\sqrt{T} = \sqrt{10}\, e_1^* \otimes e_1^* + \sqrt{5}\, e_2^* \otimes e_2^* = \sqrt{10} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$+ \sqrt{5} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \sqrt{10} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + \sqrt{5} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Side note: positive (definite) matrices define norms or semi-norms:

positive $T \geq 0$ $\quad \|u\|_T = \sqrt{u \cdot Tu}$ $\quad$ semi-norm

pos. def $T > 0$ $\quad \|u\|_T = \sqrt{u \cdot Tu}$ $\quad$ norm

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Idea:
One way to create a positive(definite) tensor

(3)

Assume F is a second order tensor. Then F^tF is a positive tensor and it's a positive definite tensor if det $F \neq 0$

$$C = F^tF$$

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$$u \cdot Cu = u \cdot F^t F u = u \cdot F^t (Fu) = \underbrace{Fu}_{v} \cdot \underbrace{Fu}_{v} = |v|^2 \geq 0$$

so $C = F^t F$ is positive

Let's check the definiteness

$u \cdot Cu = 0$ is $u = 0$ necessarily or not

$$\left.\begin{aligned} u \cdot Cu &= |v|^2 = 0 \\ v &= Fu \end{aligned}\right\} \Rightarrow v = Fu = 0$$

if $\det F \neq 0 \Rightarrow F^{-1} F u = 0 \Longrightarrow u = 0$

F^{-1} exists

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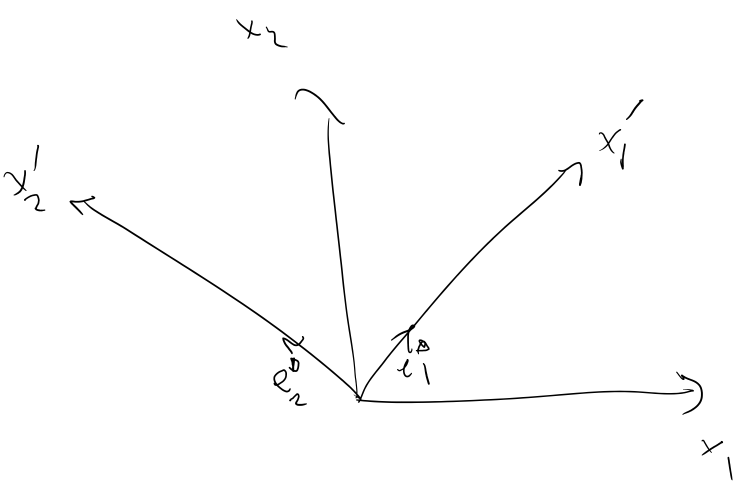
(4) if tensor C is positive & sym we can define a unique positive square root

$$[C]^* = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$$

c_i eigenvalues of C



$$C \geq 0 \Longleftrightarrow c_i \geq 0$$

$$[\sqrt{C}]^* = \begin{bmatrix} \sqrt{c_1} & 0 & 0 \\ 0 & \sqrt{c_2} & 0 \\ 0 & 0 & \sqrt{c_3} \end{bmatrix}$$

$\sqrt{9} = \pm 3$

we want the ≥ 0 root, then only 3 is chosen root, then only 3 is chosen

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Theorem 112 (Polar Decomposition Theorem) *Let* $\mathbf{F} \in \operatorname{Inv} \mathcal{V}$. *Then* $\exists$ *a unique pair of tensors* $\mathbf{U}, \mathbf{V} \in$ Psym *and a unique* $\mathbf{R} \in \operatorname{Orth} \mathcal{V} \ni$

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}.$$

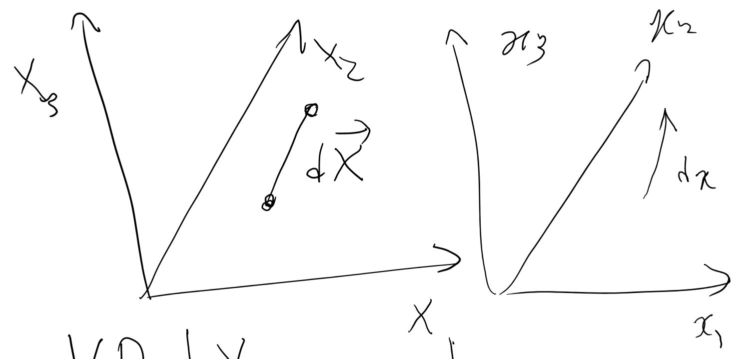
Moreover, $\det \mathbf{R} = +1$ *or* $\det \mathbf{R} = -1$, *depending as* $\det \mathbf{F} > 0$ *or* < 0.

Motivation: We'll use this equation later ...

segment $\vec{dX}$ is mapped to

segment $\vec{dx} = F\,\vec{dX}$

$$\vec{dx} = \underbrace{R}_{\text{rotation}}\underbrace{U}_{\text{straining component}}\overline{dX} = \underbrace{V}_{\text{straining operation}}\underbrace{R}_{\text{rotate}}\,dX$$



$$F = \underbrace{R}_{\text{orthogonal tensor}}\underbrace{U}_{U \geq 0 \;\&\; \text{sym}}$$

what would R & U be?

$$F^tF = (RU)^t RU = (U^tR^t)(RU) = U^t\underbrace{(R^tR)}_{\substack{\text{if } R \text{ is orthogonal} \\ I}}U = U^tIU$$

$$= U^tU = UU = U^2$$

U is sym

$U^2 = F^tF \implies U = \sqrt{F^tF}$

$C = F^tF \geq 0$ & sym

$C^t = (F^tF)^t = F^t(F^t)^t = F^tF = C$

since $C = F^tF \geq 0$ & sym

square root exists & is positive, symmetric itself from (1)

we just showed

$$\boxed{\begin{array}{l} F = RU \\ U = \sqrt{C}, \; C = F^tF \\ (5) \quad (*) \; R = FU^{-1} \text{ will be orthonormal} \end{array}}$$

check (*): $R^tR = (FU^{-1})^t FU^{-1} = U^{-t}F^tFU^{-1} =$

$= U^{-1}(F^tF)U^{-1} = U^{-1}(U^2)U^{-1} = I$ $\quad$ R is orthogon

U is sym

2nd part $\quad F = VR \quad$ V sym & ≥ 0

R same as above

$V = FR^{-1}$, $\quad F = RU \implies$

$V = RUR^{-1} = RUR^t$ $\quad$ (R is orthogonal)

show i) V is sym

& ii) V is pos

$V^t = V$?

$$V^t = (RUR^t)^t = (R^t)^t U^t R^t$$
$$= R \underset{U^t = U}{U} R^t = V$$

i) ✓

ii) is V positive

$\underset{\text{vector}}{x} \cdot Vx \geq 0$ want to show this

$$x \cdot Vx = x \cdot RUR^t x = x \cdot R(UR^t x)$$
$$= \underbrace{(R^t x)}_{y = R^t x} \cdot U(R^t x) = y \cdot Uy \geq 0 \quad \text{because } U \geq 0$$

Finally I want to show

$$V^2 = FF^t$$

$$V = RUR^t \Longrightarrow V^2 = (RUR^t)(RUR^t) =$$
$$RU\underbrace{(R^t R)}_{I}UR^t = RUIUR^t = RU^2R^t = FU^{-1}(U^2)(FU^{-1})^t$$

$$= FU^{-1}U^2U^{-t}F^t = FU^{-1}U^2U^{-1}F^t = FF^t$$

U: sym

$F = RU$ $\qquad \det F = (\det R)\underset{\geq 0}{(\det U)}$

if $\det F > 0 \Longrightarrow \det R > 0 \Longrightarrow \det R = 1$ rotation

$\det F < 0 \Longrightarrow \det R < 0 \Longrightarrow \det R = -1$ rotation plus reflection

Knowing $R^t R = I$

$\det F < 0 \implies \det R < 0 \implies \det R = -1$ reflection

Knowing $R^t R = I$
$\implies \det R = \pm 1$

summary

$\boxed{\begin{array}{l} F \text{ is given} \\ \quad - \; F = RU = VR \\ \qquad \bullet \; U = \sqrt{C}, \quad C = F^t F \qquad \text{both are sym \& positive} \\ \qquad \circ \; V = \sqrt{B}, \quad B = F F^t \\ \quad R = F U^{-1} = V^{-1} F \quad \text{is orthogonal} \\ \qquad - \; \det F > 0 \implies \det R = 1 \quad \text{rotation} \\ \qquad - \; \det F < 0 \implies \det R = -1 \end{array}}$

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