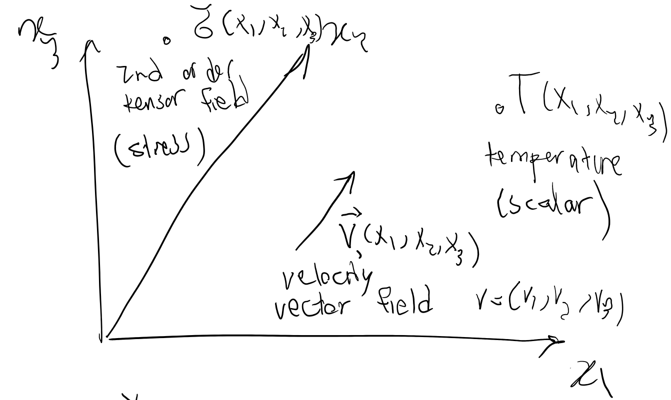
CM2023/10/16

Monday, October 16, 2023 9:45 AM

Tensor fields:
At each spatial location, we deal with a tensor



Examples of uses of tensor fields

Balance law

$$\int_{\partial \omega} \sigma \cdot n \, ds + \int_{\omega} \rho b \, dV = 0$$

$$\int_{\partial \omega} \boxed{\nabla \cdot \sigma} \, dV + \int_{\omega} \rho b \, dV = 0$$

What are $\underset{\text{grad}}{\nabla}$, $\underset{\text{div}}{\nabla\cdot}$, $\underset{\text{curl}}{\nabla\times}$ in 2D, 3D?

Heat equation $\quad q = -k \underbrace{\nabla T}_{\text{grad}}$

curl $\quad \dot{D} - \underbrace{\nabla \times H}_{\text{curl}} + J = 0 \quad$ from Maxwell's eqns

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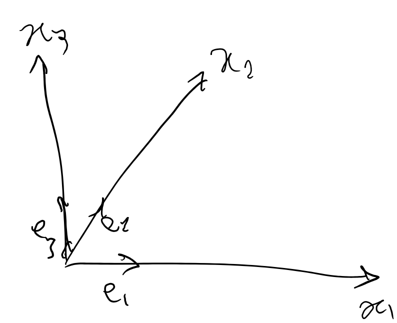
Gradient

T is an m-th order tensor field:

$$T = T_{i_1 \cdots i_m} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_m}$$

$$\underbrace{\nabla T}_{m+1 \text{ order tensor}} = T_{i_1 \cdots i_m, j} \, e_{i_1} \otimes \cdots \otimes e_{i_m} \otimes e_j \qquad (1)$$

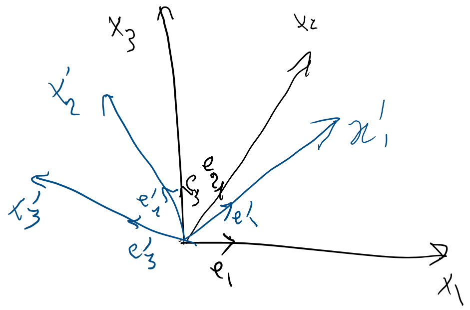
where $\quad T_{i_1 \cdots i_m, j} = \dfrac{\partial T_{i_1 \cdots i_m}(x_1, x_2, x_3)}{\partial x_j}$



for an orthogonal coordinate system, fixed in space for an orthogonal coordinate system, fixed in space

Since definition (1) is coordinate-dependent (using components of T in a given coordinate system) we need to demonstrate that it's a tensor

Example, a 2nd order tensor:



$$(\nabla T)_{ijk} = \frac{\partial T_{ij}(x_1, x_2, x_3)}{\partial x_3} \quad \text{in } (e_1, e_2, e_3)$$

$$(\nabla T)'_{mnp} = \frac{\partial T'_{mn}(x_1', x_2', x_3')}{\partial x_p'} \quad (\text{in } e_1', e_2', e_3')$$

I need to show

$$(\nabla T)'_{mnp} = Q_{mi} Q_{nj} Q_{pk} (\nabla T)_{ijk}$$

$$(\nabla T)'_{mnp} = \frac{\partial T'_{mn}}{\partial x_p'} = \frac{\partial \overbrace{Q_{mi} Q_{nj}}^{\text{not depending on } x} T_{ij}(x_1, x_2, x_3)}{\partial x_p'} = Q_{mi} Q_{nj} \frac{\partial T_{ij}(x_1, x_2, x_3)}{\partial x_p'}$$

$$\left(\text{note} \quad T'_{mn} = Q_{mi} Q_{nj} T_{ij} \right)$$

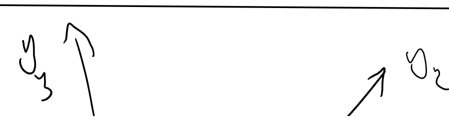
$$= Q_{mi} Q_{nj} \underbrace{\overbrace{\frac{\partial T_{ij}(x_1, x_2, x_3)}{\partial x_k}}^{\nabla T_{ijk}} \frac{\partial x_k}{\partial x_p'}}_{\frac{\partial T_{ij}}{\partial x_p'}} = Q_{mi} Q_{nj} \frac{\partial x_k}{\partial x_p'} \nabla T_{ijk}$$

$$\left(\text{note} \quad \underbrace{x_k = Q_{rk} x_r'}_{\text{from coordinate transformation}} \rightarrow \frac{\partial x_k}{\partial x_p'} = \frac{\partial Q_{rk} x_r'}{\partial x_p'} = Q_{rk} \left(\frac{\partial x_r'}{\partial x_p'} \right) = Q_{rk} \delta_{rp} = Q_{pk} \right.$$

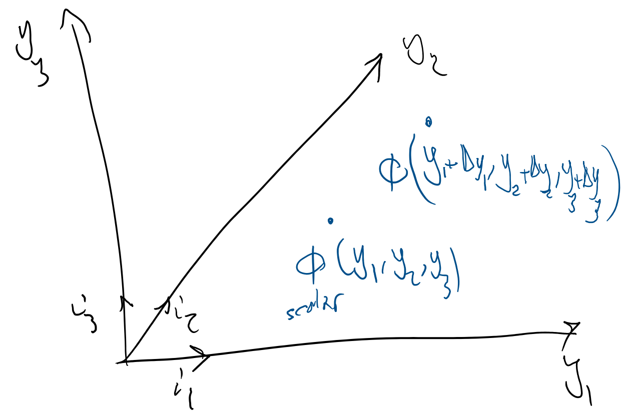
$$\Rightarrow \quad \boxed{(\nabla T)'_{mnp} = Q_{mi} Q_{nj} Q_{pk} \nabla T_{ijk}} \quad (2)$$

We demonstrated that the gradient of a second order tensor is a 3rd order tensor (components transform as 3rd order tensor). Same applies to any other tensor order.

Interpretation of the gradient operator:



Interpretation of the gradient operator:



(y_1, y_2, y_3) is the Cartesian coordinate system

Given: change of location from $\vec{y}$ to $y+\Delta y$

Question:

$$\underbrace{\Delta\phi}_{\text{total increment}} = \phi(y+\Delta y) - \phi(y)$$

$$\phi(y_1+\Delta y_1, y_2+\Delta y_2, y_3+\Delta y_3) = \phi(y_1,y_2,y_3) + \underbrace{\left(\frac{\partial\phi}{\partial y_1}\Delta y_1 + \frac{\partial\phi}{\partial y_2}\Delta y_2 + \frac{\partial\phi}{\partial y_3}\Delta y_3\right)}_{\delta\phi\text{ : first increment of }\phi}$$

$\frac{\partial\phi}{\partial y_i}\Delta y_i$

$$+\underbrace{\left(\frac{1}{2}\frac{\partial^2\phi}{\partial y_i\partial y_j}\Delta y_i\Delta y_j\right)}_{\delta^2\phi\text{ 2nd increment},\ldots}$$

$$\underbrace{\Delta\phi}_{\text{total variation}} = \phi(y+\Delta y) - \phi(y) = \delta\phi + \delta^2\phi + \delta^3\phi + \ldots = \underbrace{\begin{bmatrix}\frac{\partial\phi}{\partial y_1} & \frac{\partial\phi}{\partial y_2} & \frac{\partial\phi}{\partial y_3}\end{bmatrix}\begin{bmatrix}\Delta y_1\\ \Delta y_2\\ \Delta y_3\end{bmatrix}}_{\delta\phi}$$

$$\underbrace{+\frac{1}{2}\begin{bmatrix}\Delta y_1 & \Delta y_2 & \Delta y_3\end{bmatrix}\overbrace{\begin{bmatrix}\phi_{,11} & \phi_{,12} & \phi_{,13}\\ \phi_{,21} & \phi_{,22} & \phi_{,23}\\ \phi_{,31} & \phi_{,32} & \phi_{,33}\end{bmatrix}}^{\text{Hessian}}\begin{bmatrix}\Delta y_1\\ \Delta y_2\\ \Delta y_3\end{bmatrix}}_{\delta^2 y} + \underbrace{\text{H.O.T}}_{\text{higher order terms}}$$

$$\phi_{,ij} = \frac{\partial^2\phi}{\partial y_i\partial y_j}$$

$$\boxed{\begin{array}{l}\text{as } \Delta y \to 0 \qquad \Delta\phi = \delta\phi + \delta^2\phi \ldots \approx \delta\phi \\ \text{scalar} \qquad \delta\phi = \underbrace{\text{grad}\,\phi}_{\text{vector}}\,\Delta y\end{array}}$$

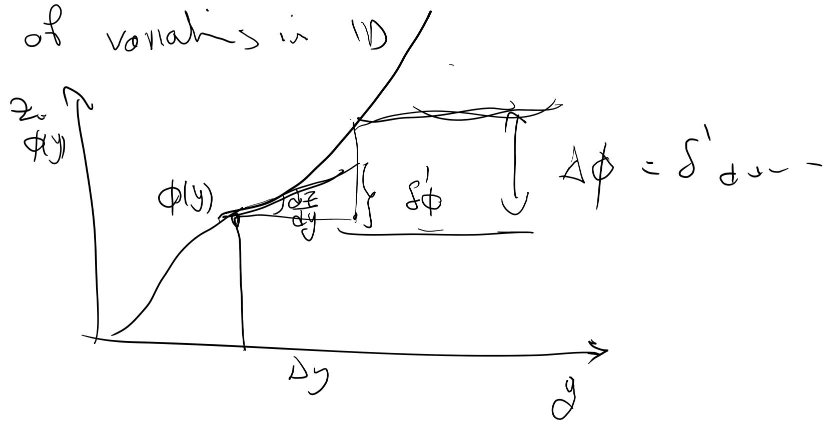
scalar $\delta\phi = \underbrace{\text{grad}\,\phi}_{\text{vector}}\ \Delta y$

vector $\delta v = \underbrace{(\text{grad}\ v)}_{\text{2nd order tensor}}\ \Delta y$

$\delta T = \underbrace{(\text{grad}\ T)}_{\text{3rd order}}\ \Delta y$

(3)

geometric interpretation of variations in 1D



We can do a Taylor's expansion of a vector and observe:

$$v = v_1(y_1, y_2, y_3)e_1 + v_2e_2 + v_3e_3 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}(y_1, y_2, y_3)$$

$$\Delta v = (\text{grad}\ v)\,\Delta y + \text{HOTs}$$

$$\text{grad}\ v = \begin{bmatrix} v_{1,1} & v_{1,2} & v_{1,3} \\ v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,1} & v_{3,2} & v_{3,3} \end{bmatrix}$$

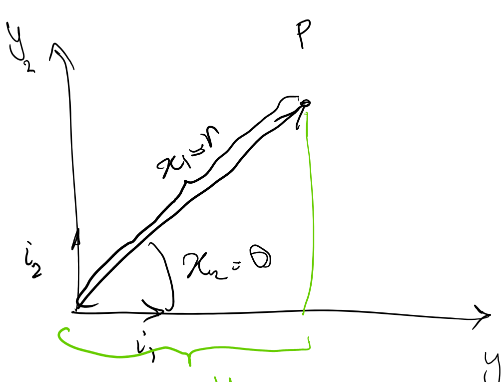
this matches

$$(\nabla T)_{i_1 \dots i_n j} = \frac{\partial T_{i_1 \dots i_n}}{\partial y_j}$$

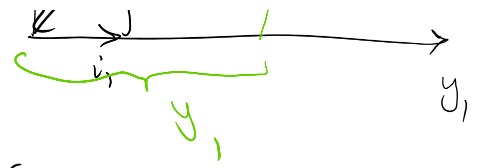
Orthonormal curvilinear coordinate systems:

$$\begin{cases} y_1 = y_1(x_1, x_2) \\ y_2 = y_2(x_1, x_2) \end{cases}$$

def of curvilinear coordinate system & relation between x & y



system &
relation between x&y



Example: Polar coordinate system: $x_1 = r$, $x_2 = \theta$

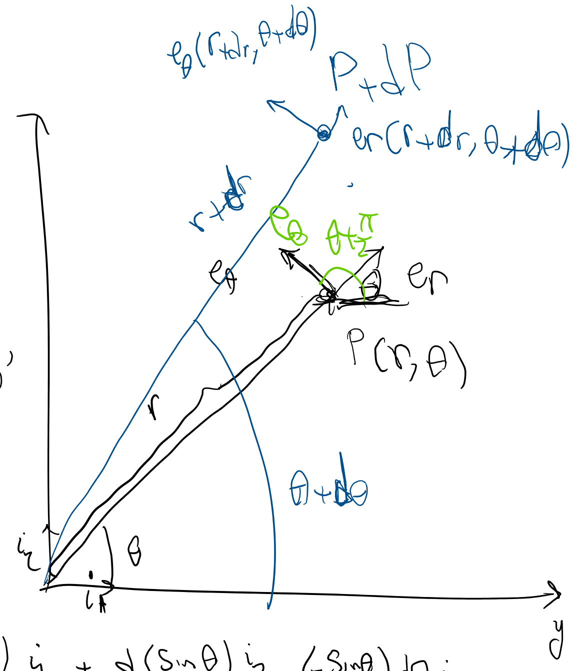
$$\begin{cases} y_1 = \overset{r}{\widetilde{x_1}} \cos \overset{\theta}{\widetilde{x_2}} \\ y_2 = x_1 \sin x_2 \end{cases}$$

How do we calculate the gradient in the polar coordinate system?

$\vec{P} = r\vec{e}_r$

④ $\underbrace{d\vec{P}}_{\text{change of position } P} = d(re_r) = (dr)e_r + r(de_r)$

$(ab)' = a'b + ab'$



formulas for e_r & e_θ

$e_r = (\cos\theta)\, i_1 + (\sin\theta)\, i_2 \rightarrow de_r = d(\cos\theta)\, i_1 + d(\sin\theta)\, i_2 = (-\sin\theta)\, d\theta\, i_1 + (\cos\theta)\, d\theta\, i_2$

$\Longrightarrow de_r = (-\sin\theta\, i_1 + \cos\theta\, i_2)\, d\theta = (d\theta)\, e_\theta$

$e_\theta = \cos(\theta + \frac{\pi}{2})\, i_1 + \sin(\theta + \frac{\pi}{2})\, i_2 = -\sin\theta\, i_1 + \cos\theta\, i_2$

$\rightarrow de_\theta = d(-\sin\theta)\, i_1 + d(\cos\theta)\, i_2 = -(\cos\theta)\, d\theta\, i_1 - (\sin\theta)\, d\theta\, i_2 = -d\theta\, (\underbrace{\cos\theta\, i_1 + (\sin\theta)\, i_2}_{e_r})$

$de_\theta = -(d\theta)\, e_r$

eq 4

$d\vec{P} = d(re_r) =$

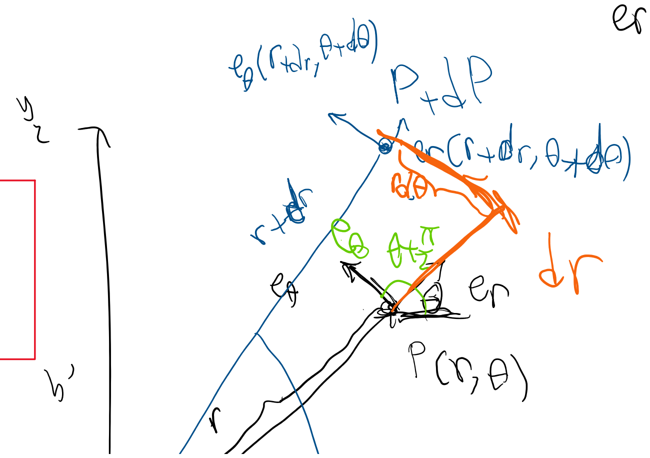
$(dr)\, e_r + r\, de_r$

$(dr)e_r + r((d\theta) \ldots$

⑤

$$\boxed{\begin{aligned} de_r &= (d\theta)\, e_\theta \\ de_\theta &= (-d\theta)\, e_r \end{aligned}}$$

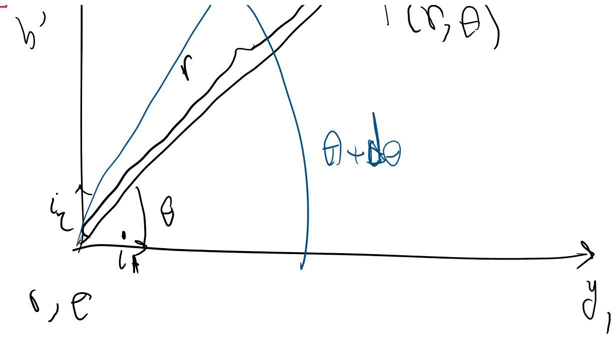
b'



$(d\theta)e_r + r\,d\theta$

$= (dr)e_r + r(d\theta\, e_\theta)$

$\Rightarrow (6)\ \boxed{\vec{dP} = (dr)e_r + (r\,d\theta)e_\theta}$



ϕ is a scalar field as a function of r, θ

$\phi(r,\theta)$ $\qquad d\phi = \frac{\partial\phi}{\partial r}dr + \frac{\partial\phi}{\partial\theta}d\theta = \frac{\partial\phi}{\partial r}\underbrace{(dr)}_{dP_r} + \left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)\underbrace{(r\,d\theta)}_{dP_\theta}$

eqn 6 $\qquad \vec{dP} = \underbrace{dr}_{(dP)_r}\,e_r + \underbrace{(r\,d\theta)}_{(dl)_\theta}\,e_\theta$

$$\longrightarrow \quad d\phi = \underbrace{\left[\frac{\partial\phi}{\partial r} \quad \frac{1}{r}\frac{\partial\phi}{\partial\theta}\right]}_{\text{grad } \phi \text{ in polar coordinate}} \begin{pmatrix} (\vec{dP})_r \\ (\vec{dP})_\theta \end{pmatrix}$$

$$d\phi = (\text{grad}\,\phi)\ \vec{dP}$$

$$\nabla\phi = \left[\frac{\partial\phi}{\partial r} \quad \frac{1}{r}\frac{\partial\phi}{\partial\theta}\right] \quad \text{in polar coordinate}$$

$V = v_r e_r + v_\theta e_\theta$ vector

(8) $$\nabla V = \begin{bmatrix} V_{r,r} & \frac{V_{r,\theta} - V_\theta}{r} \\ V_{\theta,r} & \frac{V_{\theta,\theta} + V_r}{r} \end{bmatrix}$$

$\nabla \cdot V = V_{r,r} + \frac{V_{\theta,\theta} + V_r}{r}$

$V_{r,r} = \frac{\partial V_r}{\partial r}$

$dV = (\nabla V)\,dP$

Demonstration of gradient equation for polar coordinate, for a vector

$V = v_r e_r + v_\theta e_\theta \rightarrow dv = d(v_r e_r) + d(v_\theta e_\theta) = (dv_r) e_r + v_r (de_r) + (dv_\theta) e_\theta + v_\theta (de_\theta) =$

$(v_{r,r} dr + v_{r,\theta} d\theta) e_r + v_r (e_\theta d\theta) + (v_{\theta,r} dr + v_{\theta,\theta} d\theta) e_\theta + v_\theta (-e_r d\theta) =$

$\left(v_{r,r}\, dr + \left(\frac{v_{r,\theta} - v_\theta}{r}\right) r d\theta\right) e_r + \left(v_{\theta,r} + \frac{v_{\theta,\theta} + v_r}{r} (r d\theta)\right) e_\theta$, Note $dP_r = dr$, $dP_\theta = r d\theta$

$\Rightarrow dv = (dv)_r e_r + (dv_\theta) e_\theta = \left(v_{r,r} dP_r + \left(\frac{v_{r,\theta} - v_\theta}{r}\right) dP_\theta\right) e_r + \left(v_{\theta,r} dP_r + \frac{v_{\theta,\theta} + v_r}{r} dP_\theta\right) e_\theta$

$$\begin{bmatrix} dv_r \\ dv_\theta \end{bmatrix} = \underbrace{\begin{bmatrix} v_{r,r} & \frac{v_{r,\theta} - v_\theta}{r} \\ v_{\theta,r} & \frac{v_{\theta,\theta} + v_r}{r} \end{bmatrix}}_{\text{So this is the grad operator in polar coordinate}} \begin{bmatrix} dP_r \\ dP_\theta \end{bmatrix}$$

note $dV = (\text{grad}\, V)\, \underbrace{dP}_{\text{change in position}}$

How do we calculate divergence & curl : T is m'th order

$$\underset{m-1 \text{ order}}{\nabla \cdot T} = T_{i_1 \ldots i_{m-1} j, j}\, e_{i_1} \otimes \cdots \otimes e_{i_{m-1}}$$

last two entries are contracted

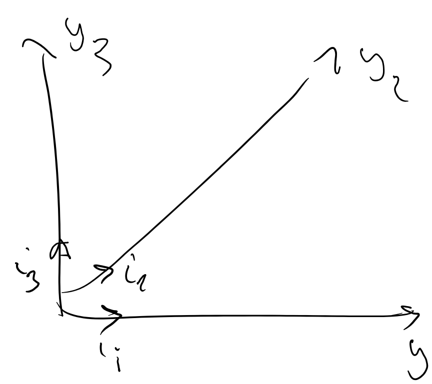
$m = 1$ $\quad \nabla \cdot T = T_{i,i} = \text{trace}(\nabla T)$

— often defined for vector

$\text{curl}\, V = ax(\text{skew}\, \nabla T)$

Examples in Cartesian coordinate

$v = (v_1, v_2, v_3)$



$$\nabla V = \begin{bmatrix} v_{1,1} & v_{1,2} & v_{1,3} \\ v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,1} & v_{3,2} & v_{3,3} \end{bmatrix}$$

$$\lfloor V_{3,1} \quad V_{3,2} \quad V_{3,3} \rfloor$$

$$\nabla \cdot V = \mathrm{trace}(\nabla V) = V_{1,1} + V_{2,2} + V_{3,3} = V_{i,i}$$

$$\mathrm{curl}\, V = \mathrm{ax}(\mathrm{skew}\, \nabla V) = \underbrace{\nabla \times V = \det \begin{bmatrix} i_1 & i_2 & i_3 \\ \frac{\partial}{\partial y_1} & \frac{\partial}{\partial y_2} & \frac{\partial}{\partial y_3} \\ V_1 & V_2 & V_3 \end{bmatrix}}_{\nabla \times V}$$

$$\mathrm{Curl}\, V = \begin{bmatrix} V_{3,2} - V_{2,3} \\ V_{1,3} - V_{3,1} \\ V_{2,1} - V_{1,2} \end{bmatrix}$$

Martinec_Zdenek_Charles_U_Prague_Martinec-ContinuumMechanics.pdf
Please read appendix C