CM2023/10/23

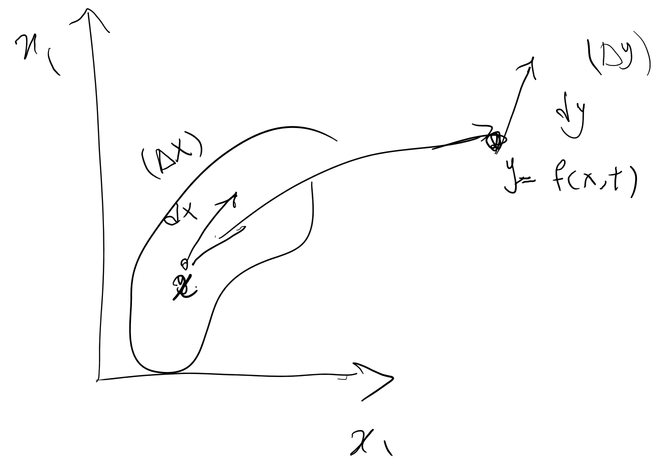
Monday, October 23, 2023 9:42 AM

From last time:

$$\boxed{\begin{array}{c}\vec{\Delta y} = F\,\vec{\Delta x} \\ \vec{dy} = F\,\vec{dx}\end{array}}$$

0

↓ 2nd order tensor



$$F = \nabla_{y/x} = \nabla_x \underbrace{f(x,t)}_{y}$$

$$\begin{cases} y_1 = 1.1\,x_1 \\ y_2 = 1.5\,x_2 \\ y_3 = 1.2\,x_1 + 1.7\,x_3 \end{cases} \qquad F = \nabla_{y/x} = \begin{bmatrix} 1.1 & 0 & 0 \\ 0 & 1.5 & 0 \\ 1.2 & 0 & 1.7 \end{bmatrix}$$

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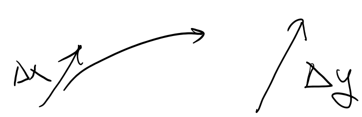
1. $|\Delta y| = ?$ $\qquad \vec{u}, \quad |u| = \sqrt{u \cdot u}$

$$\left.\begin{array}{l} |\Delta y| = \sqrt{\Delta y \cdot \Delta y} \\ \Delta y = F\,\Delta x \end{array}\right\} \rightarrow |\Delta y| = \sqrt{(F\,\Delta x)\cdot(F\Delta x)} = \sqrt{(F^t F \Delta x)\cdot \Delta x}$$

1

$$\boxed{\begin{array}{l} C = F^t F \quad \text{right Cauchy-Green deformation tensor} \\ |\Delta y| = \sqrt{\Delta x \cdot C \Delta x} \\ \underline{\text{Length}} \text{ of } \Delta y \end{array}}$$



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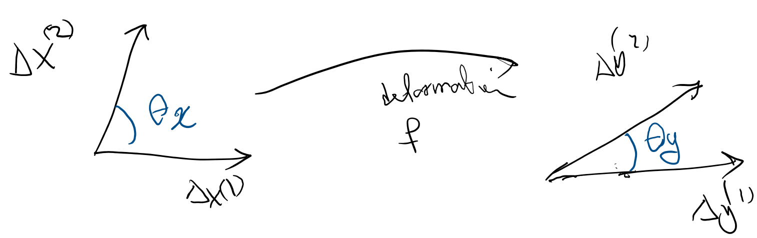
2. (Change of) angle

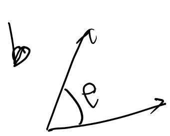
$$\Delta X^{(1)} = \begin{bmatrix} \Delta x_1^{(1)} \\ \Delta x_2^{(1)} \\ \Delta x_3^{(1)} \end{bmatrix}$$

↓ is a vector



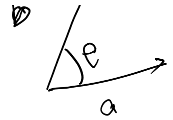
at time t

(*) recall



$a \cdot b = |a|\,|b| \cos\theta$

(*) $\cos\theta_y = \dfrac{\Delta y^{(1)} \cdot \Delta y^{(2)}}{|\Delta y^{(1)}|\ \ldots}$

(*) recall  $a \cdot b = |a|\,|b| \cos\theta$

(*) $\cos\theta_y = \dfrac{\Delta y^{(1)} \cdot \Delta y^{(2)}}{|\Delta y^{(1)}|\,|\Delta y^{2}|}$

$\cos\theta_y = \dfrac{\Delta y^{(1)} \circ \Delta y^{(2)}}{|\Delta y^{(1)}|\,|\Delta y^{2}|}$

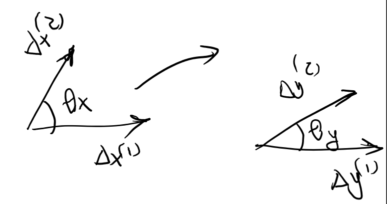
(1) $\Rightarrow |\Delta y^{(i)}| = \sqrt{\Delta y^{(i)} \cdot C\,\Delta y^{(i)}}$ no summation on i

(0) $\Rightarrow \Delta y^{(i)} = F \Delta x^{(i)}$

$\Rightarrow \cos\theta_y = \dfrac{F\Delta x^{(1)} \cdot F\Delta x^{(2)}}{\sqrt{\Delta x^{(1)} \cdot C \Delta x^{(1)}}\,\sqrt{\Delta x^{(2)} \cdot C\Delta x^{(2)}}}$

$\rightarrow$

(2) $$\boxed{\cos\theta_y = \frac{\Delta x^{(2)} \cdot C\,\Delta x^{(1)}}{\sqrt{\Delta x^{(1)} \cdot C\Delta x^{(1)}}\,\sqrt{\Delta x^{(2)} \cdot C\Delta x^{(2)}}} \qquad C = F^t F}$$

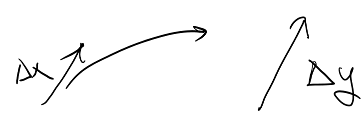


$$\boxed{\begin{aligned} \vec{\Delta y} &= F\,\vec{\Delta x} \\ \vec{dy} &= F\,\vec{dx} \end{aligned}}$$ (0)

(1) $C = F^t F$ right Cauchy-Green deformation tensor

$|\Delta y| = \sqrt{\Delta x \circ C \Delta x}$

Length of Δy



$|\Delta y|$ & θ_y will be used to define normal & shear strains

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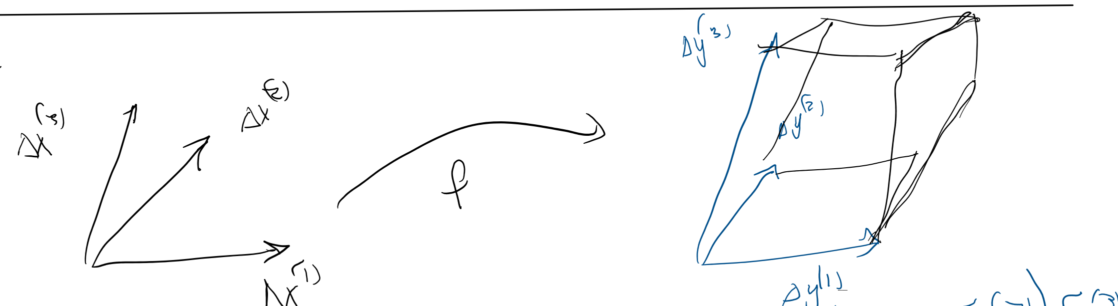
3. (Change of) Volume



$\Delta V_x = \underbrace{(\Delta x^{(1)} \times \Delta x^{(2)}) \cdot \Delta x^{(3)}}_{\text{triple product}} =$

similarly $\Delta V_y = (F\Delta x^{(1)} \times F\Delta x^{(2)}) \cdot F\Delta x^{(3)}$

$\Delta V_y =$

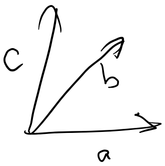
(i) ... (ii) ...

triple product

(i) $\det \begin{vmatrix} \Delta x_1^{(1)} & \Delta x_2^{(1)} & \Delta x_3^{(1)} \\ \Delta x_1^{(2)} & \Delta x_2^{(2)} & \Delta x_3^{(2)} \\ \Delta x_1^{(3)} & \Delta x_2^{(3)} & \Delta x_3^{(3)} \end{vmatrix}$

(ii) $\Delta V_y = \det \begin{vmatrix} \Delta y_1^{(1)} & \Delta y_2^{(1)} & \Delta y_3^{(1)} \\ \cdot & \text{---} & \text{---} \\ \cdot & \cdot & \Delta y_3^{(3)} \end{vmatrix}$

Recall



$$\text{volume}(a,b,c) = (a \times b)\cdot c = \epsilon_{ijk}\, a_i\, b_j\, c_k$$

iii)

Apply iii to ii)

$$\Delta V_y = \epsilon_{ijk}\, \Delta y_i^{(1)}\, \Delta y_j^{(2)}\, \Delta y_k^{(3)}$$

$$\Delta y^{(m)} = F\, \Delta x^{(m)}$$

$$\Delta y_i^{(1)} = F_{ia}\, \Delta x_a^{(1)}$$

$$\Delta y_j^{(2)} = F_{jb}\, \Delta x_b^{(2)}$$

$$\Delta y_k^{(3)} = F_{kc}\, \Delta x_c^{(3)}$$

$$\Rightarrow \Delta V_y = \epsilon_{ijk} \left(F_{ia}\Delta x_a^{(1)}\right)\left(F_{jb}\Delta x_b^{(2)}\right)\left(F_{kc}\Delta x_c^{(3)}\right)$$

$$= \underbrace{\left(\epsilon_{ijk} F_{ia} F_{jb} F_{kc}\right)} \Delta x_a^{(1)} \Delta x_b^{(2)} \Delta x_c^{(3)}$$

$$= \epsilon_{abc} \det F \;\; \Delta x_a^{(1)} \Delta x_b^{(2)} \Delta x_c^{(3)}$$

$$= \underbrace{\left(\epsilon_{abc} \Delta x_a^{(1)} \Delta x_b^{(2)} \Delta x_c^{(3)}\right)}_{\text{using ii} \;\; \Delta V_x} \det F$$

3

$$\boxed{\begin{array}{l} \Delta V_y = J\, \Delta V_x \\ J := \det F \quad \text{Jacobian of deformation} \\ J \text{ is a scalar} \end{array}}$$

Recall

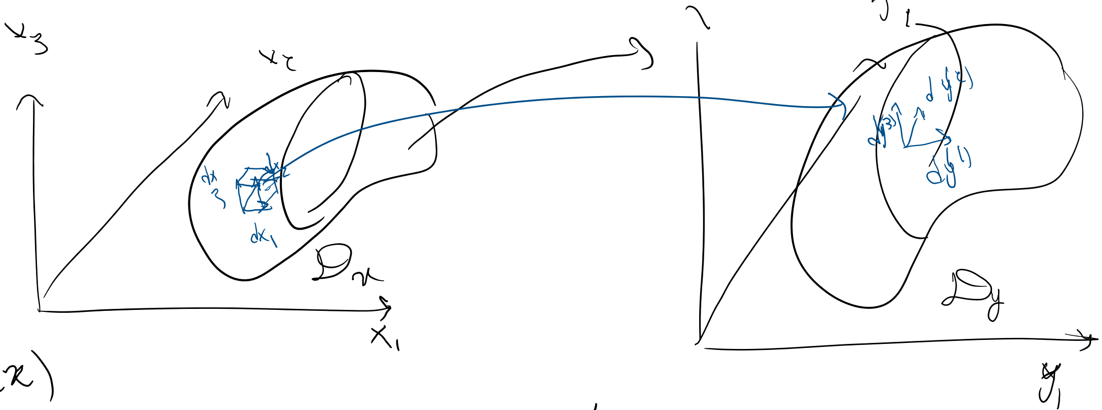
Definition 72 *Let* $\overset{0}{\mathcal{B}}$ *be an open, bounded, regular region of a Euclidean point space* $\mathcal{E}$. *A* **deformation f** *is a mapping (function) of points in* $\overset{0}{\mathcal{B}}$ *onto another open region of* $\mathcal{E}$ *with the properties*

1. *f is one-to-one; i.e.,* $\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{y}) \Rightarrow \mathbf{x} = \mathbf{y}\ \forall\ \mathbf{x}, \mathbf{y} \in \overset{0}{\mathcal{B}}$,
2. $\mathbf{f} \in C^2(\overset{0}{\mathcal{B}})$, $\mathbf{f}^{-1} \in C^2(\mathbf{f}(\overset{0}{\mathcal{B}}))$,
3. $\det \nabla \mathbf{f}(\mathbf{x}) > 0\ \forall\ \mathbf{x} \in \overset{0}{\mathcal{B}}$.

3. $\det \nabla \mathbf{f}(\mathbf{x}) > 0 \ \forall\, \mathbf{x} \in \overset{0}{\mathcal{B}}$.

The notation $\mathbf{f}(\overset{0}{\mathcal{B}})$ *refers to the mapped region, which is called the image of the set* $\overset{0}{\mathcal{B}}$ *under* $\mathbf{f}$.

Use of J in this context is not new



$y = f(x)$

$$V_y = \int_{D_y} dV_y$$

$$= \int_{D_x} J \, dV_x$$

$$J = \det(\nabla_x y) = \det \nabla_x f(x) = \det F$$

$$dV_y = J \, dV_x$$

In Balance laws:

$$\int_{D_y} f(y,t)\, dV_y = \int_{D_x} f(y(x,t),t)\,(J\, dV_x)$$

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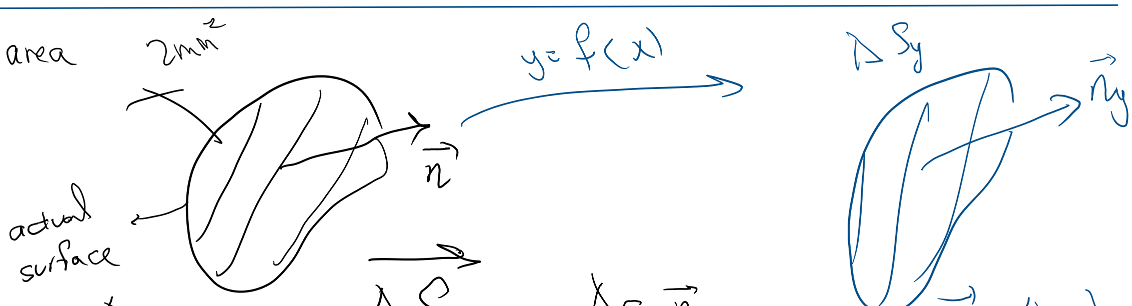
4. (Change of) area



ΔS_x

$$\underbrace{\overrightarrow{\Delta S_x}}_{\substack{\text{a covector} \\ \text{of size } \Delta S_x \\ \text{\& direction } \vec{n}}} = \underset{2\,mm^2}{\Delta S_x} \cdot \underset{\vec{n}}{\vec{n}}$$

$$\overrightarrow{\Delta S_y} = (\Delta S_y)\, n_y$$

of size ΔS_x

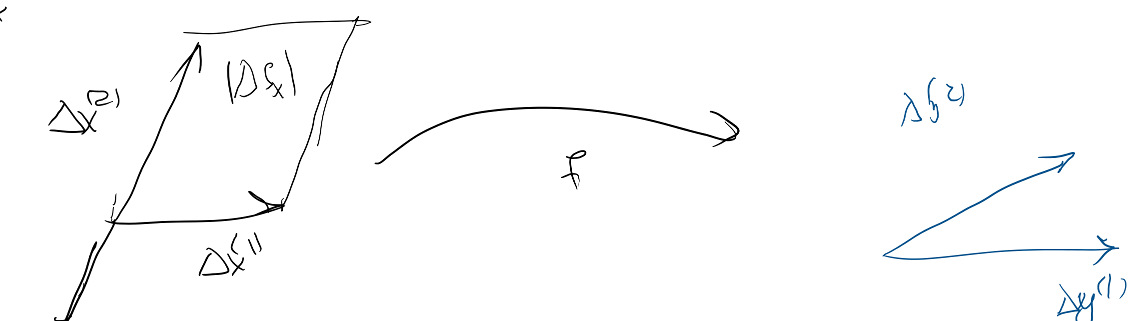
& direction $\vec{n}$

From here, we'll drop the vector notation from the surface differential as always we work with the oriented surface (surface area * normal)

we're looking for sth like

$$\vec{\Delta S_y} = \underbrace{\left(\quad\right)}_{\text{2nd order tensor}} \vec{\Delta S_x}$$

proof



$$\vec{\Delta S_x} = \Delta x^{(1)} \times \Delta x^{(2)} \qquad \vec{\Delta S_y} = \Delta y^{(1)} \times \Delta y^{(2)}$$

we need to use a trick to come up with a 3rd vector & use det formulas

$$\vec{\Delta S_y} = (\Delta S_y)_1 e_1 + (\Delta S_y)_2 e_2 + (\Delta S_3) e_3$$

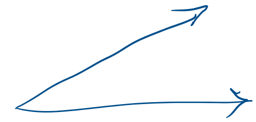
$$\Rightarrow \quad \Delta S_{y_1} = (\vec{\Delta S_y}) \cdot e_1, \ldots$$

$$\Rightarrow (\vec{\Delta S_y})_i = (\vec{\Delta S_y}) \cdot e_i = (\Delta y^{(1)} \times \Delta y^{(2)}) \cdot e_i$$

$$= \left((F \Delta x^{(1)}) \times (F \Delta x^{(2)})\right) \cdot \underbrace{(F F^{-1})}_{I} e_i$$

$\Delta y^{(2)}$

$= ((\;\;\Delta x\;\;) \times (\;\;\Delta x\;\;)) \cdot (\underbrace{\;\;\;}_{I}\;\;)$



$$= (F\Delta x^{(1)}) \times (F\Delta x^{(2)}) \cdot F(F^{-1} e_i)$$

$$= \det F \, (\Delta x^{(1)} \times \Delta x^{(2)}) \cdot F^{-1} e_i$$

like the volume of p.e.d

$\Delta y^{(1)}$

$\vec{\Delta S_y} = \Delta y^{(1)} \times \Delta y^{(2)}$

$\Delta y^{(\alpha)} = F \Delta x^{(\alpha)}$

$$= \det F \; F^{-t}\Big(\underbrace{(\Delta x^{(1)} \times \Delta x^{(2)}) \cdot e_i}_{(\underbrace{\Delta S_x}_{V})_i}\Big)$$

$V = v_i e_i$

$v_i = V \cdot e_i$

$$= \det F \; F^{-t} (\vec{\Delta S_x})_i$$

$$\underbrace{(\Delta S_y)_i}_{A_i} = \underbrace{\det F \; F^{-t} (\Delta S_x)_i}_{B_i}$$

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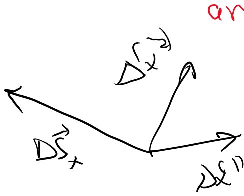
new covectors "surfaces" map

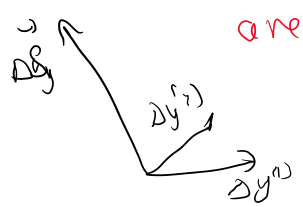
(4)

$$\vec{\Delta S_y} = \det F \; F^{-t} \, \vec{\Delta S_x}$$

new surface area

reference surface area





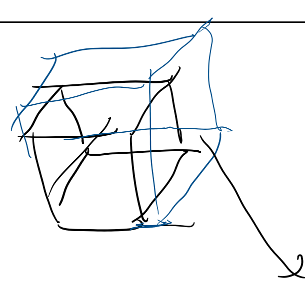
Compare with

$\Delta y = F \, \Delta x$ how vectors are mapped

$y_1 = 1.1 x_1$

$y_2 = 1.1 x_2$

$y_3 = 1.1 x_3$



$$F = \begin{bmatrix} 1.1 & 0 & 0 \\ 0 & 1.1 & 0 \\ 0 & 0 & 1.1 \end{bmatrix}$$

$dy = F dx$ 1.1

(10% increase)

$dV_y = (\det F)\, dV_x$

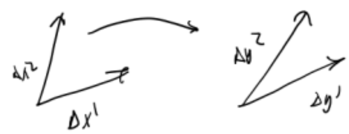
1.1^3

1.13

$dS_y = dS_x\,(1.1)^2$

21%

Another proof of this:



$$\Delta \vec{S}_y = \Delta y^1 \times \Delta y^2 = e_{ijk}\, \Delta y^1_i\, \Delta y^2_j\, e_k$$

$$\Delta y^1 = F\Delta x^1 \Rightarrow \Delta y^1_i = F_{im}\Delta x^1_m$$

$$\Delta y^2 = F\Delta x^2 \Rightarrow \Delta y^2_j = F_{jn}\Delta x^2_n$$

$$\Delta S_y = \epsilon_{ijk}\, \overbrace{(F_{im}\Delta x^1_m)}^{\Delta y^1_i}\, \underbrace{(F_{jn}\Delta x^2_n)}_{\Delta y^2_j}\, e_k =$$

$$\frac{\det F}{\det F}\left(\epsilon_{ijk}F_{im}F_{jn}\right)\Delta x^1_m\, \Delta x^2_n\, e_k$$

$$\epsilon_{ijk}F_{im}F_{jn} = \det F\, \epsilon_{mnp}\, F^{-1}_{pk}$$

$$\Delta S_y = \left(\epsilon_{mnp}F^{-1}_{pk}\det F\right)\Delta x^1_m\, \Delta x^2_n\, e_k = \det F\, \underbrace{\left(\epsilon_{mnp}\Delta x^1_m\, \Delta x^2_n\right)}_{(\Delta x^1 \times \Delta x^2)_p}\, F^{-1}_{pk}\, e_k$$

$$= \det F\; F^{-T}_{kp}\,(\Delta x^1 \times \Delta x^2)_p\; e_k$$

$$= \det F\,\left(F^{-T}\Delta S_x\right)_k e_k = \det F\, F^{-T}\Delta \vec{S}_x$$