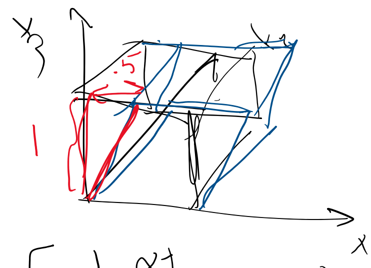
CM2023/10/25

Wednesday, October 25, 2023 9:47 AM

Matlab test file that will be shared with you

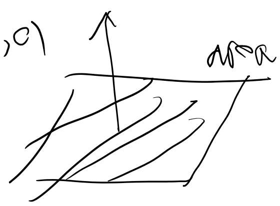
```
% displacement factor is alpha * t where alpha is a factor and t is time
alphat = 5e-1;
F = [1 alphat 0; 0 1 0; 0 0 1];
```

$$\begin{cases} y_1 = x_1 + \alpha t x_2 \\ y_2 = x_2 \\ y_3 = x_3 \end{cases}$$

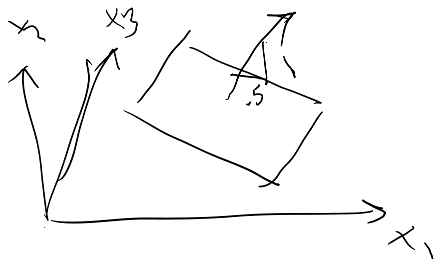


$$F = \nabla y_{/x} = \begin{bmatrix} 1 & \alpha t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$dS_x = (0,1,0)$ area $= 1$

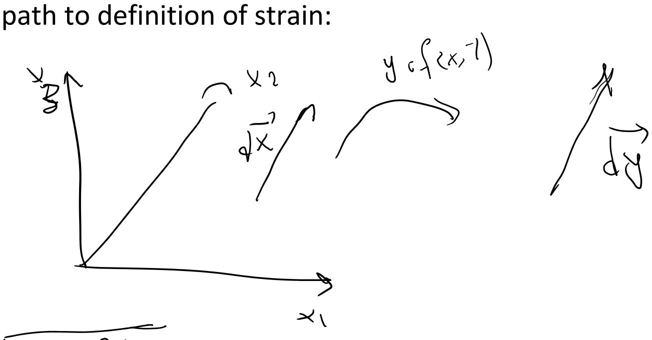


$dS_y = (0.5, 1, 0)$



Further discussion on the right Cauchy-Green tensor C and path to definition of strain:

$$C = F^t F$$



$$\left.\begin{aligned} |dy| &= \sqrt{dx \cdot C dx} \\ U^2 &= C \end{aligned}\right\} \rightarrow |dy| = \sqrt{dx \cdot U^2 dx} = \sqrt{dx \cdot U (U dx)} = \sqrt{(\underbrace{U^t}_{= U} dx) \cdot (U dx)}$$

both C & U are symmetric (& also pos. def.)

$$|dy| = \sqrt{(U dx) \cdot (U dx)} = \sqrt{|U dx|^2} \Rightarrow |dy| = U |dx|$$

$$\boxed{\text{from } |dy| = \sqrt{dx \cdot C dx} \text{ we obtain } |dy| = \underbrace{U}_{\text{stretch tensor}} |dx| \qquad U = \sqrt{C}}$$

(1)

Recall C symmetric $\longrightarrow$ 3 orthonormal directions for eigenvectors

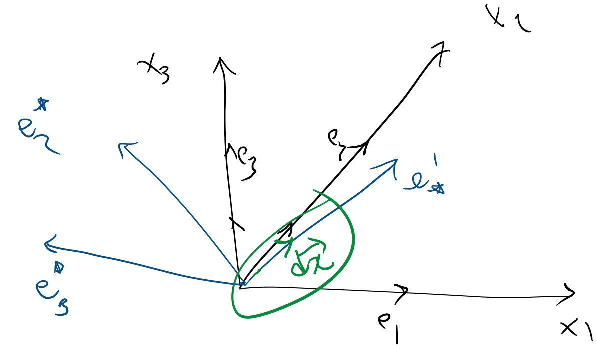
$C > 0 \longrightarrow c_1, c_2, c_3$ eigen values > 0

x_1

$C > 0 \rightarrow c_1, c_2, c_3$ eigen values > 0

C_i e_i^* eigenvectors of C

$e_i^* \cdot e_j^* = \delta_{ij}$



$$[C]^* = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$$

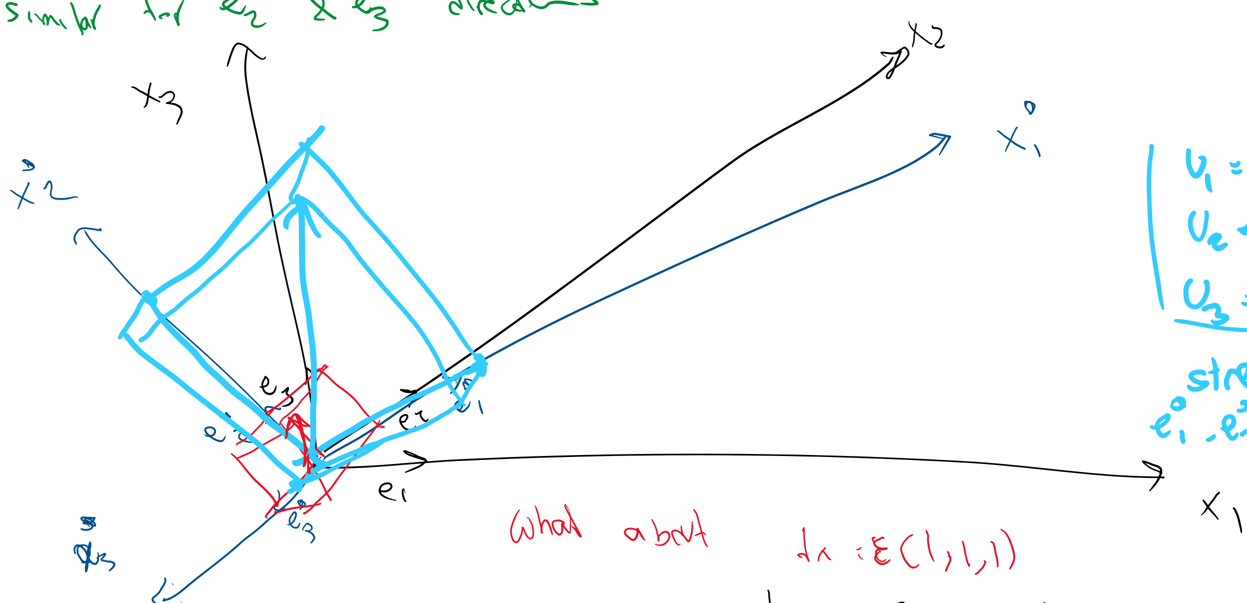
$$[U]^* = \begin{bmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{bmatrix} \qquad u_i = \sqrt{c_i}$$

$U = \sqrt{C}$

assume $dx = \varepsilon e_1^*$ from ① $dy = U dx = U \varepsilon e_1^* = \varepsilon (U e_1^*)$ $\varepsilon \ll 1$

$= \varepsilon (u_1 e_1^*) = u_1 (\varepsilon e_1^*) = u_1 \, dx$ $\boxed{\vec{dy} = u_1 dx}$

similar for e_2^* & e_3^* directions



$u_1 = 2$
$u_2 = 3$
$u_3 = .5$

stretches in e_1^*, e_2^*, e_3^* directions

What about $dx = \varepsilon(1,1,1)$

$dy = \varepsilon(u_1, u_2, u_3)$ $(= U dx)$

$dy \not\parallel dx$ in general

except when $dx \parallel e_1^*$, or e_2^*, or e_3^*

Relation to polar decomposition theorem

Relation to polar decomposition theorem

Recall $F = (\nabla y_x)$, $\det F = J > 0$

we have
$$\begin{cases} F = RU = VR \\ U = \sqrt{C},\ C = F^t F \\ V = \sqrt{B},\ B = F F^t \end{cases}$$

Definition 80 *Let the deformation gradient* $\mathbf{F} = \nabla \mathbf{f}$ *of the deformation* $\mathbf{f}$ *of* $\overset{0}{\mathcal{B}}$ *have the polar decompostion*

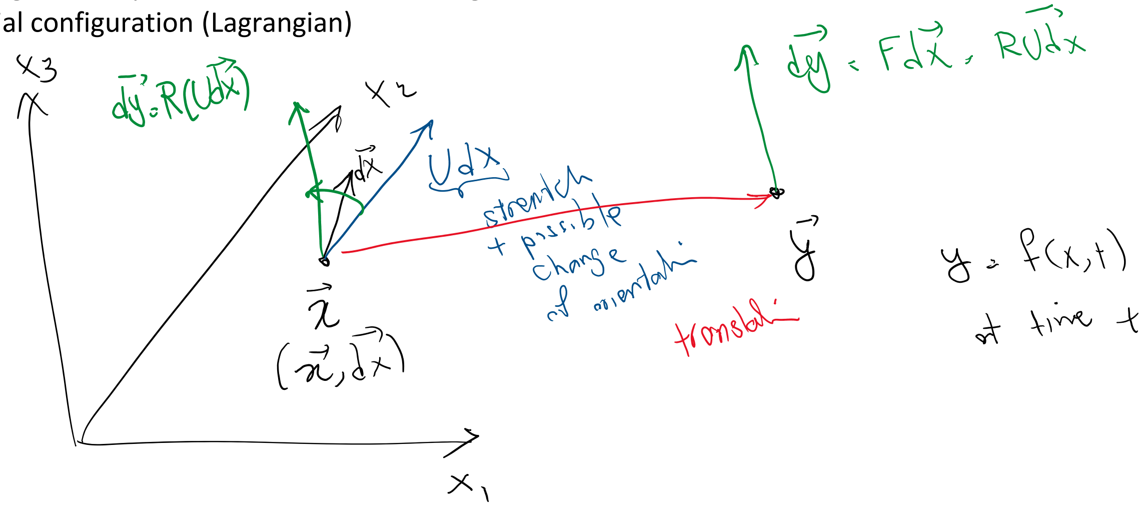
$$\mathbf{F}(\mathbf{x}) = \mathbf{R}(\mathbf{x})\mathbf{U}(\mathbf{x}) = \mathbf{V}(\mathbf{x})\mathbf{R}(\mathbf{x})$$

$\forall\ \mathbf{x} \in \overset{0}{\mathcal{B}}$, *where* $\mathbf{U}(\mathbf{x}), \mathbf{V}(\mathbf{x}) \in \mathrm{Psym}$ *and* $\mathbf{R}(\mathbf{x}) \in \mathrm{Orth}\ \mathcal{V}^+$. *The following terminology is standard.*

- $\mathbf{R}(\mathbf{x})$ — *the* rotation tensor *at* $\mathbf{x}$;
- $\mathbf{U}(\mathbf{x})$ — *the* right stretch tensor *at* $\mathbf{x}$;
- $\mathbf{V}(\mathbf{x})$ — *the* left stretch tensor *at* $\mathbf{x}$;
- $\mathbf{C}(\mathbf{x}) = \mathbf{F}^t(\mathbf{x})\mathbf{F}(\mathbf{x})$ — *the* right Cauchy-Green deformation tensor *at* $\mathbf{x}$;
- $\mathbf{B}(\mathbf{x}) = \mathbf{F}(\mathbf{x})\mathbf{F}^t(\mathbf{x})$ — *the* left Cauchy-Green deformation tensor *at* $\mathbf{x}$.

Understanding the components of a motion of a segment:

A) Referential configuration (Lagrangian)



$y = f(x,t)$ at time t

$$\left.\begin{aligned} \vec{dy} &= F\,\vec{dx} \\ F &= RU \end{aligned}\right\} \Rightarrow dy = RU\vec{dx} = R(\underbrace{U\vec{dx}}_{\text{stretch tensor}})$$

1. stretch U: real deformation $(\vec{x}, U\vec{dx})$ ①

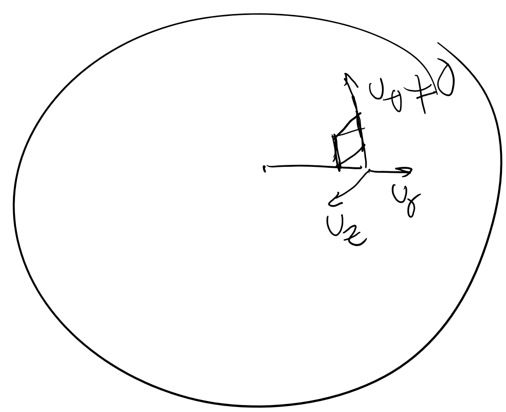
②

1. stretch U: real deformation $(\vec{x}, U d\vec{x})$ ① ②

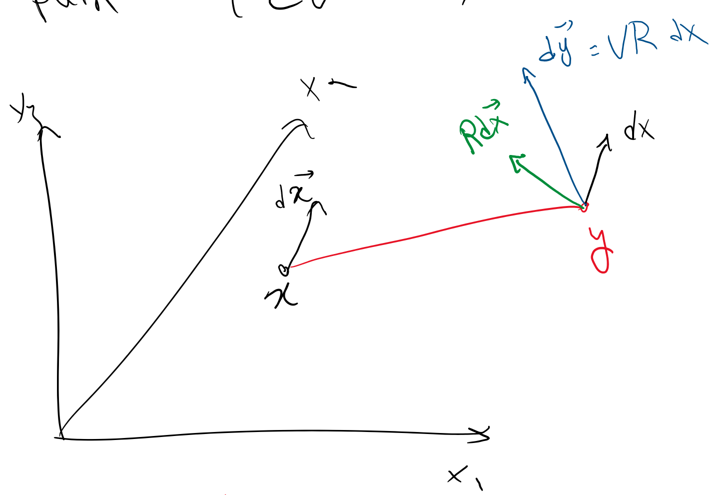
2. rotation R $(\vec{x}, d\vec{y})$ $d\vec{y} = R U d\vec{x}$

3. translation $\vec{x} \to \vec{y}$ $(\vec{y}, d\vec{y})$

→ rigid motion parts



Left path (Eulerian) $F = RU = VR$



① translation $(x, dx) \to (\vec{y}, d\vec{x})$

② Rotation R $(y, R dx)$ $d\vec{y} = F d\vec{x} = V R\, dx = V(R d\vec{x})$

③ Stretch (Left stretch tensor V) $(y, d\vec{y})$ $d\vec{y} = V(R d\vec{x})$

$U = \sqrt{C}$, $C = F^t F$ $U \neq V$

$V = \sqrt{B}$, $B = F F^t$

$U = VC$, $C = \ldots$

$V = \sqrt{B}$, $B = FF^t$

$U \neq V$

$\underline{RU \neq UR}$ for matrices in general $AB \neq BA$

$RU = VR$

But the principal stretches of U & V are the same

Assume vector a is an eigenvector of U

$U a = \lambda a$ — eigenvector for a $\longrightarrow$ $\underbrace{RU}_{F} a = \lambda Ra \Rightarrow Fa = \lambda \underbrace{(Ra)}$

$\Rightarrow \underbrace{(VR)}_{F} a = \lambda Ra \longrightarrow V(Ra) = \lambda (Ra)$

$$\boxed{\begin{array}{l} \underset{\text{eigen val of } U}{\lambda} \; \underset{\text{eigenvec of } U}{a}: \quad Ua = \lambda a \Longrightarrow V \underbrace{(Ra)} = \lambda \underbrace{(Ra)} \\ V \text{ has the same eigenvalues with Rotated eigenvectors} \end{array}}$$

(2)

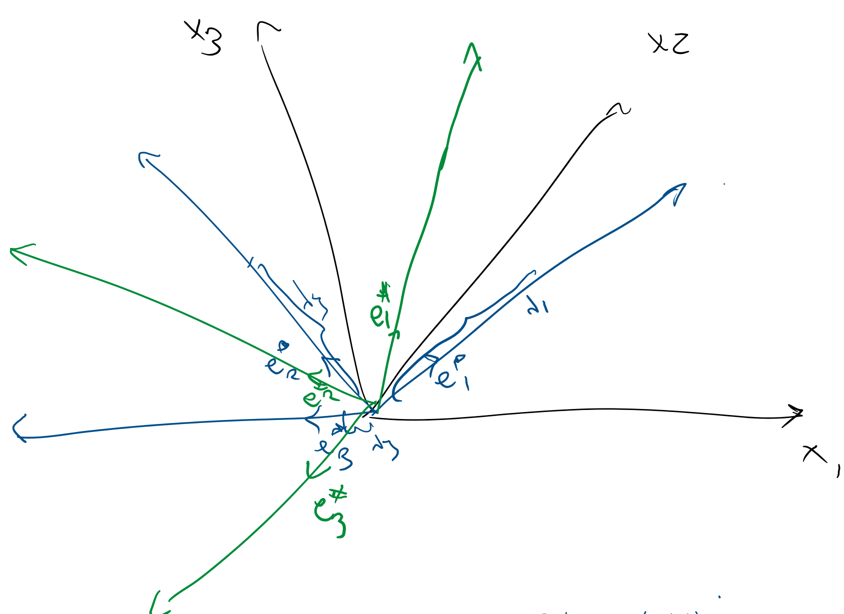
$a = e_1^*$

e_1^*, e_2^*, e_3^* are eigenvectors of U

$$[U]^* = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{bmatrix}$$

$\lambda_1, \lambda_2, \lambda_3$ eigen values of U

$$[V]^\# = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$$

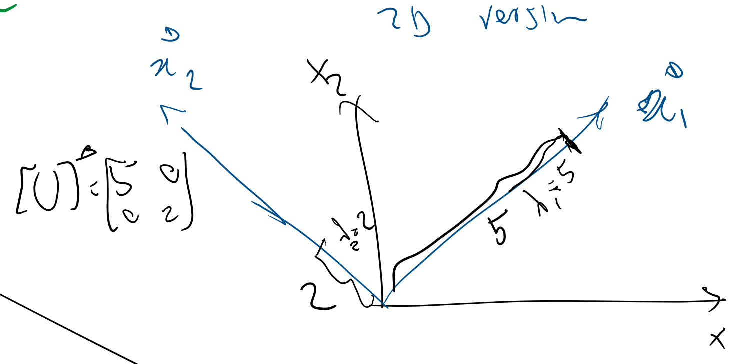


2D version

$[V^\circ] = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$

$e_1^\#, e_2^\#, e_3^\#$

2D version



$e_i^\# = R\, e_i^\circ$

V is U rotated by R

$VR = F \rightarrow V = FR^t = RUR^t$

$\underbrace{F}_{RU}$

$U = R^t VR$

U is rotation of V by R^t

3

Theorem 128 *Let* **f** *be a deformation on* $\overset{\circ}{\mathcal{B}}$. *Then*

1. $\mathbf{C} = \mathbf{U}^2$, $\mathbf{B} = \mathbf{V}^2$,
2. $\mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^t$, $\mathbf{U} = \mathbf{R}^t\mathbf{V}\mathbf{R}$;
3. $\mathbf{B} = \mathbf{R}\mathbf{C}\mathbf{R}^t$, $\mathbf{C} = \mathbf{R}^t\mathbf{B}\mathbf{R}$.

Infinitesimal theory

W & E commute

$H = \nabla u_{/x} = \overbrace{E + W = W + E}$

$E = \dfrac{\nabla u + (\nabla u)^t}{2}$ $\quad W = \dfrac{\nabla u - (\nabla u)^t}{2}$

U or V $\quad$ R (−I)

minus identity

$\underbrace{\text{U or V}}_{\text{deformation}}$ $\quad \underbrace{\text{R}(-I)}_{\text{rotation}}$

But

R & U do not commute

$RU \neq UR$

Definition of G, E, and W:

$$F = \nabla y_{/X} = \nabla (u+x)_{/x} = \underbrace{\nabla u_{/x}}_{H} + \underbrace{\nabla x_{/x}}_{I}$$

$$F = H + I$$

$$C = F^t F$$

$$= (H+I)^t (H+I) = H^t H + H^t + H + I$$

$$\boxed{\begin{aligned} G &= \frac{1}{2}(C - I) \quad \text{Green St. Venant strain tensor} \\ &= \frac{1}{2} H^t H + \frac{H^t + H}{2} = \frac{1}{2} H^t H + E \\ E &= \frac{H + H^t}{2} \quad \text{infinitesimal strain tensor} \quad E = sym(H) \\ W &= \frac{H - H^t}{2} = \text{rotation} \quad W = skew(H) \end{aligned}}$$

$$H_{ij} = \frac{\partial u_i (X, t)}{\partial x_j} \qquad H = \nabla u_{/x}$$