CM2023/10/30

Monday, October 30, 2023 9:47 AM

Continue from last time:

$$G = \frac{1}{2}(C - I)$$

Green St-Venant tensor

$$G_{ij} = \frac{1}{2}(C_{ij} - \delta_{ij})$$

$$C = F^t F \rightarrow C_{ij} = (F^t F)_{ij} = (F^t)_{im} F_{mj} = F_{mi} F_{mj}$$

$$F = H + I \rightarrow F_{mi} = H_{mi} + \delta_{mi}$$

$\Rightarrow$

$$C_{ij} = (H_{mi} + \delta_{mi})(H_{mj} + \delta_{mj}) = H_{mi} H_{mj} + H_{mj}\delta_{mi} + H_{mi}\delta_{mj} + \delta_{mi}\delta_{mj}$$

$$C_{ij} = H_{mi} H_{mj} + H_{ij} + H_{ji} + \delta_{ij}$$

$\Rightarrow$

$$\Longrightarrow \quad G_{ij} = \frac{1}{2}(C_{ij} - \delta_{ij})$$

①

$$G = \frac{1}{2}(C - I) = \frac{1}{2}(H + H^t + H^t H) = E + \frac{1}{2} H^t H$$

(finite) large deformation gradient strain

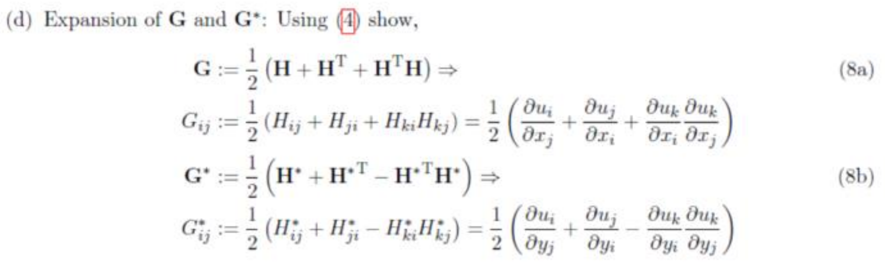
$$E = \frac{H + H^t}{2}$$

infinitesimal deformation gradient strain

$$G_{ij} = \frac{1}{2}(H_{ij} + H_{ji}) + \underbrace{\frac{1}{2} H_{mi} H_{mj}}_{\text{higher order term relative to } E}$$

$$E = \frac{1}{2}(H_{ij} + H_{ji})$$

HW6:



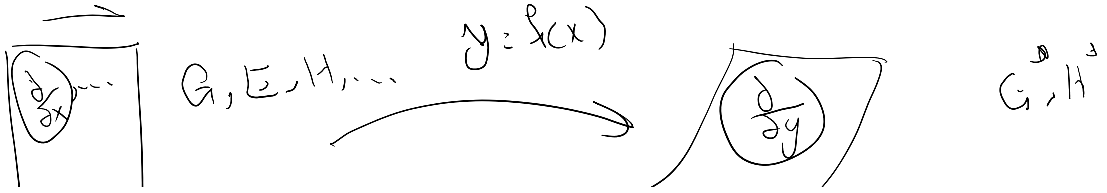
(d) Expansion of **G** and **G***: Using (4) show,

$$\mathbf{G} := \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^{\mathrm{T}} + \mathbf{H}^{\mathrm{T}}\mathbf{H}\right) \Rightarrow \tag{8a}$$

$$G_{ij} := \frac{1}{2}\left(H_{ij} + H_{ji} + H_{ki}H_{kj}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)$$

$$\mathbf{G}^* := \frac{1}{2}\left(\mathbf{H}^* + \mathbf{H}^{*\mathrm{T}} - \mathbf{H}^{*\mathrm{T}}\mathbf{H}^*\right) \Rightarrow \tag{8b}$$

$$G^*_{ij} := \frac{1}{2}\left(H^*_{ij} + H^*_{ji} - H^*_{ki}H^*_{kj}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial y_j} + \frac{\partial u_j}{\partial y_i} - \frac{\partial u_k}{\partial y_i}\frac{\partial u_k}{\partial y_j}\right)$$



solid mechanics

— solid mechanics for <u>some</u> large deformation problems

— Fluid Mechanics

G

∇

$\partial/\partial x$

G°

∇°

$\partial/\partial y$

$F = RU$

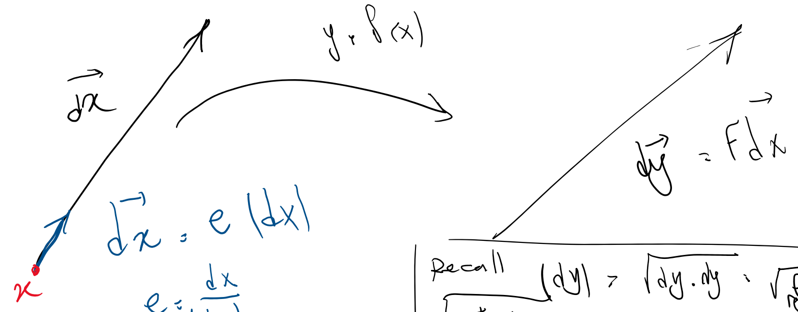
$C = F^{T}F \;,\; U = \sqrt{C}$

$F = VR$

$B = FF^{t} \quad V = \sqrt{B}$

Definition of strains

1. Normal strain



$\vec{dx} = e\,|dx|$

$e = \frac{dx}{|dx|}$

from before

$|dy| = \sqrt{dx \cdot C\,dx}$

Recall $|dy| = \sqrt{dy \cdot dy} = \sqrt{F dx \cdot F dx}$

$\sqrt{F^{t}F dx \cdot dx} = \sqrt{C\,dx \cdot dx}$

②

Strain along e @ base location x is defined as

$$\varepsilon(x,e) = \frac{\overbrace{|dy| - |dx|}^{\text{change of length}}}{\underbrace{|dx|}_{\text{original length}}}$$

$e = \frac{dx}{|dx|}$

for small $|dx|$

$|dx| \ll 1$

$$\varepsilon(x,e) = \frac{\sqrt{dx\cdot C dx} - |dx|}{|dx|} = \sqrt{\frac{1}{|dx|^2}(dx\cdot C dx)} - 1 = \sqrt{\left(\frac{dx}{|dx|}\right)\cdot C\left(\frac{dx}{|dx|}\right)} - 1 = e\cdot Ce - 1$$

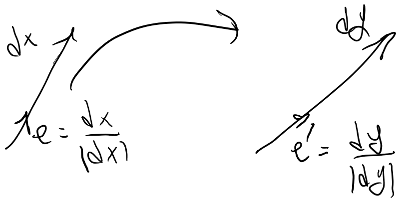
③

Strain

$$\varepsilon(\underset{\text{initial position}}{x}, \underset{\text{initial state orientation}}{e}) = \frac{|dy|-|dx|}{|dx|} = \underline{\sqrt{e\cdot \underset{U^2}{C} e} - 1} =$$

$$\sqrt{(Ue)(Ue)} - 1$$

$$= \underline{Ue - 1}$$

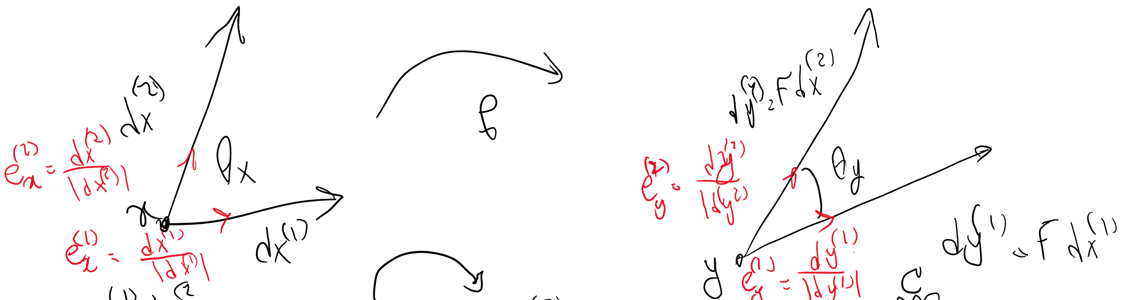


$U = \sqrt{C}$

$$\varepsilon(\underset{\text{spatial (Eulerian)}}{y, e'}) = 1 - \sqrt{e' B^{-1} e'}$$

?? check it next time

2. Shear strain:



Recall

$$\cos\theta_y = \frac{dy^{(1)}\cdot dy^{(2)}}{|dy^{(1)}||dy^{(2)}|} = \frac{F dx^{(1)}\cdot F dx^{(2)}}{\sqrt{F dx^{(1)}\cdot F dx^{(1)}}\sqrt{F dx^{(2)}\cdot F dx^{(2)}}} = \frac{dx^{(1)}\cdot \overbrace{F^t F}^{C}\, dx^{(2)}}{\sqrt{dx^{(1)}\cdot F^t F dx^{(1)}}\sqrt{dx^{(2)}\cdot F^t F dx^{(2)}}}$$

$$= \frac{dx^{(1)}\cdot C dx^{(2)}}{\sqrt{dx^{(1)}\cdot C dx^{(1)}}\sqrt{dx^{(2)}\cdot C dx^{(2)}}}$$

Let's define angle between $e_y^{(1)}, e_y^{(2)}$ having the orientations $e_x^{(1)}, e_x^{(2)}$

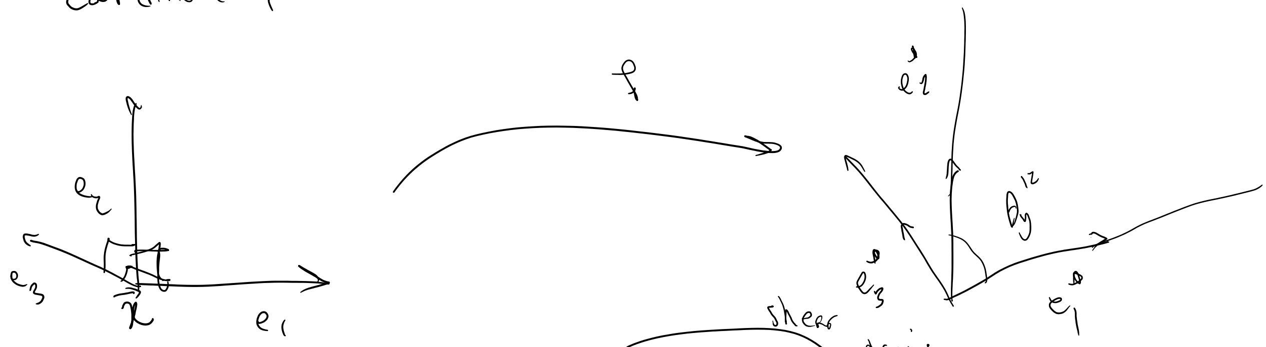
$$\cos\theta_y = \frac{(e_x^{(1)}|dx^{(1)}|)\cdot(C\; e_x^{(2)}|dx^{(2)}|)}{\sqrt{|dx^{(1)}|e_x^{(1)}\cdot C(|dx^{(1)}|e_x^{1})}\;\sqrt{(|dx^{(2)}|e_x^{2})\cdot C\,(|dx^{(2)}|e_x^{(2)})}} = \frac{e_x^{(1)}\cdot C\, e_x^{(2)}}{\sqrt{e_x^{(1)}\cdot C e_x^{(1)}}\sqrt{e_x^{(2)}\cdot C e_x^{(2)}}}$$

y

$$\sqrt{|dx^{(1)}| e_x^{(1)} \cdot C(|dx^{(1)}| e_x^{1})} \quad \sqrt{(|dx^{(2)}| e_x^{2}) \cdot C(|dx^{(2)}| e_x^{(2)})} \qquad \sqrt{e_x^{(1)} \cdot Ce_x^{(1)}}\sqrt{e_x^{(2)} \cdot Ce_x^{(2)}}$$

④

We want to define normal & shear strains for an orthonormal coordinate system:



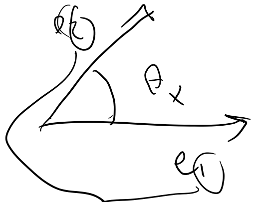
x system

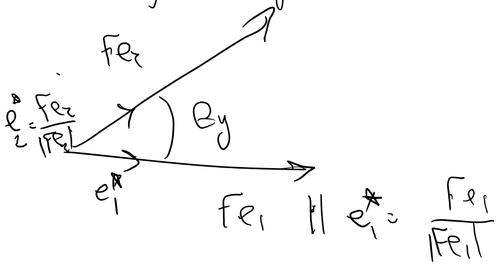
$e_i \cdot e_j = 0$

shear strains

$$\begin{bmatrix} \varepsilon(e_1) & (1,2) & (1,3) \\ & \varepsilon(e_2) & (2,3) \\ & & \varepsilon(e_3) \end{bmatrix}_{3\times3}$$

shear strains are defined as the following change of angle





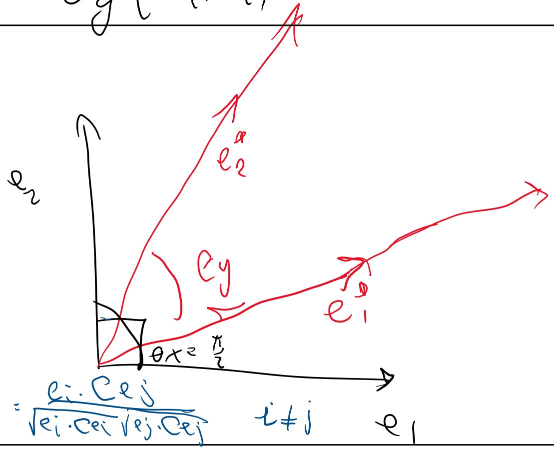
$$\gamma_{12}^{xt} = \Theta_x(e_1, e_2) - \Theta_y(e_1^*, e_2^*)$$

⑤

Always $\theta_x = \frac{\pi}{2}$

$$\gamma_{12} = \frac{\pi}{2} - \theta_y$$

$$\sin\gamma_{12} = \frac{e_1 \cdot Ce_2}{\sqrt{e_1 \cdot Ce_1}\sqrt{e_2 \cdot Ce_2}}$$



$$\sin\gamma_{ij} = \frac{e_i \cdot Ce_j}{\sqrt{e_i \cdot Ce_i}\sqrt{e_j \cdot Ce_j}} \quad i \neq j$$

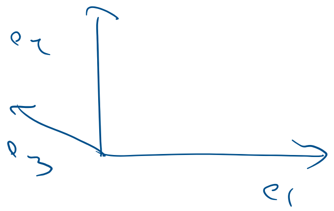
$$\sin\gamma_{12} = \sin\left(\frac{\pi}{2} - \theta_y\right) = \cos\theta_y$$

④ $\Rightarrow$ $$\cos\theta_y = \frac{e_1 \cdot Ce_2}{\sqrt{e_1 \cdot Ce_1}\sqrt{e_2 \cdot Ce_2}}$$

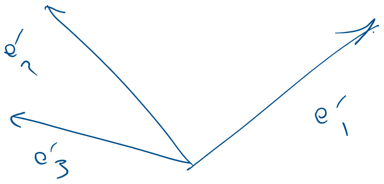
What if we put all these strains in a matrix?

$$\gamma_{ij} = \gamma_{ji} \qquad \tilde{E} \to \begin{bmatrix} \varepsilon(e_1) & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \varepsilon(e_2) & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \varepsilon(e_3) \end{bmatrix}$$

this is not a tensor



$$Q = \begin{bmatrix} e'_1 \\ \hline e'_2 \\ \hline e'_3 \end{bmatrix}$$



$$[\tilde{E}]' = [Q][\tilde{E}]Q^t$$ but this $\tilde{E}$ does not satisfy this

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If we do two things

$$\begin{bmatrix} \varepsilon(e_1) & \frac{\gamma_{12}}{2} & \frac{\gamma_{13}}{2} \\ \frac{\gamma_{12}}{2} & \varepsilon(e_2) & \frac{\gamma_{23}}{2} \\ \frac{\gamma_{13}}{2} & \frac{\gamma_{23}}{2} & \varepsilon(e_3) \end{bmatrix}$$

AND Use infinitesimal theory approximati (next)

$$\underset{\sim}{E} = \frac{H + H^t}{2} = \begin{bmatrix} u_{1,1} & \frac{u_{1,2}+u_{2,1}}{2} & \frac{u_{1,3}+u_{3,1}}{2} \\ & u_{2,2} & \frac{u_{2,3}+u_{3,2}}{2} \\ & & u_{3,3} \end{bmatrix}$$

approximations of $\varepsilon(e_1)$ $\varepsilon(e_2)$ $\varepsilon(e_3)$

approximations of $\frac{\gamma_{12}}{2}$, $\frac{\gamma_{13}}{2}$, $\frac{\gamma_{23}}{2}$

why E is a tensor?

$$\underset{\sim}{H} = \nabla u \qquad \left(H_{ij} = \frac{\partial u_i}{\partial x_j} \right)$$

$$H = \nabla_x u \qquad \left(H_{ij} = \frac{\partial u_i}{\partial x_j} \right)$$

H as a gradient of a tensor, is a tensor

$\Longrightarrow H^t$ is a tensor as H is

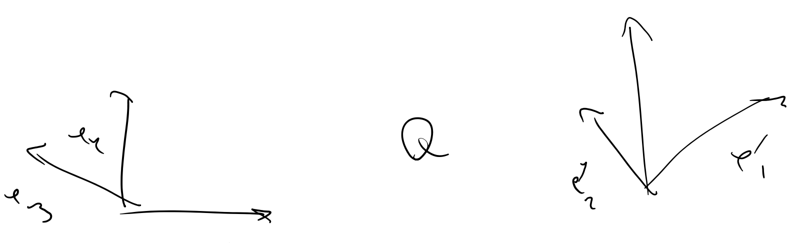
$\Big\} \Longrightarrow E = \frac{H + H^t}{2}$ is a tensor

⑥

in infinitesimal theory (discussed next)

$$E = \frac{H + H^t}{2}$$

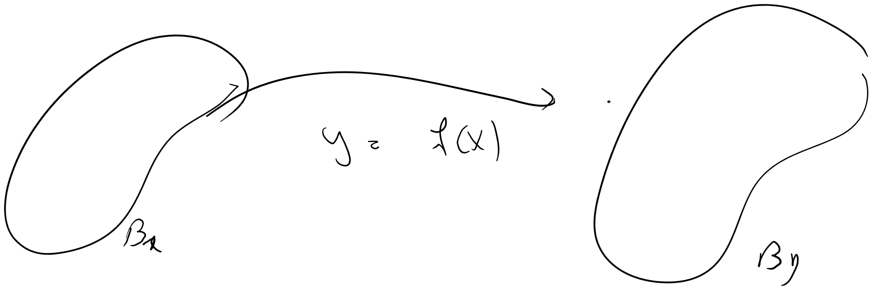
is a tensor



$$[E'] = Q[E]Q^t$$

Why the normal and the shear strains we defined can be approximated by the E tensor?

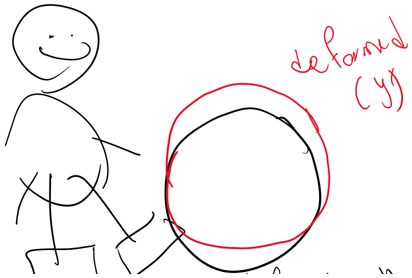
Infinitesimal deformation theory



$u = y - x$ displacement

$$dy = F\,dx \rightarrow du = (F - I)\,dx = H\,dx$$

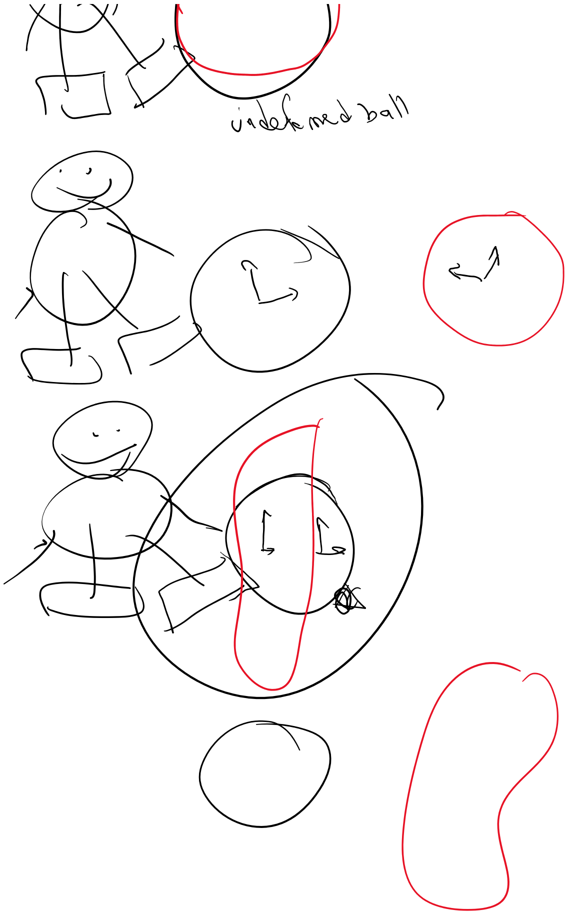
$$du = H\,dx \qquad \boxed{H = \nabla u_{/x} = F - I}$$



$\underbrace{u}_{\text{displacement}}$ small

$\underbrace{H = \nabla_x u}_{\text{deformed}}$ small

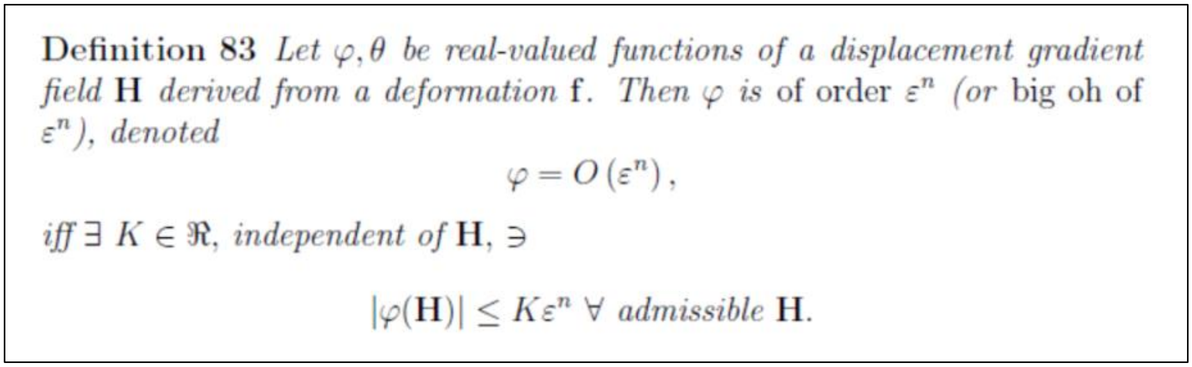
easy small strains $\Longrightarrow$



small	small	⟹ make good approximation
large challenge: tracking contacts	small	Infinitesimal theory $H = E + W$ strain, rotation
Small	large large strain	finite deformation gradient theory $F = RU = UR$
large	large	$\mathcal{E}(e) = Ue - 1$ - - -

We want to approximatate

$\mathcal{E}(e_i)$ $\frac{\gamma_{ij}}{2}$



Definition 83 *Let φ, θ be real-valued functions of a displacement gradient field* **H** *derived from a deformation* **f**. *Then φ is of order ε^n (or* big oh of *ε^n), denoted*

$$\varphi = O(\varepsilon^n),$$

iff $\exists\ K \in \Re$, independent of **H**, $\ni$

$$|\varphi(\mathbf{H})| \le K\varepsilon^n \ \forall \text{ admissible } \mathbf{H}.$$

Definition 116 *Let* $\Psi : A \subset \text{Lin}\ \mathcal{V} \rightarrow \Re$ *have the property* $\Psi(\mathbf{0}) = 0$, *and let* $\varepsilon := \|\mathbf{A}\|$, $\mathbf{A} \in A$.[10] *Then we say that* Ψ *is* little oh of ε,[11] *denoted*

$$\Psi(\mathbf{A}) = o(\varepsilon) \ \textit{as}\ \varepsilon \rightarrow 0,$$

iff

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon \neq 0}} \frac{|\Psi(\mathbf{A})|}{\varepsilon} = 0.$$