CM20231101

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Definition 83 *Let φ, θ be real-valued functions of a displacement gradient field* $\mathbf{H}$ *derived from a deformation* $\mathbf{f}$. *Then φ is* of order ε^n *(or* big oh of ε^n*), denoted*

$$\varphi = O(\varepsilon^n),$$

iff $\exists\, K \in \Re$, *independent of* $\mathbf{H}$, $\ni$

$$|\varphi(\mathbf{H})| \leq K\varepsilon^n \;\forall \text{ admissible } \mathbf{H}.$$

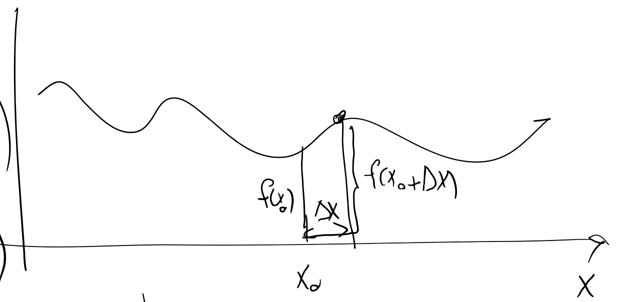
Example

$$f(x_0+\Delta x) = f(x_0) + \Delta x f'(x_0) + \left(\frac{\Delta x^2}{2} f''(x_0) + \frac{\Delta x^3}{6} f'''(x_0) + \dots\right)$$

$$\underbrace{\Delta x^2 \left(\frac{f''(x_0)}{2} + \frac{\Delta x}{6} f'''(x_0) + \dots \right)}$$

as $\Delta x \to 0$

can be neglected relative to



$O(\Delta x^2)$

there is a C such that

$$\left| \Delta x^2 \left(\frac{f''(x_0)}{2} + \dots \right) \right| \leq C \Delta x^2$$

as $\Delta x \to 0$

eg

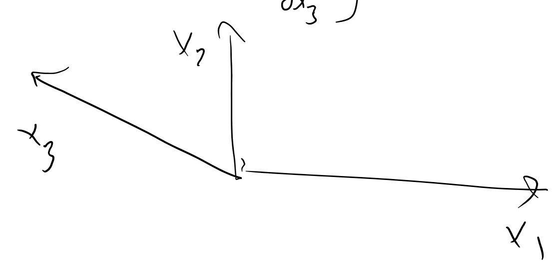
$C = |f''(x_0)|$

Use of this in infinitesimal theory:

H is very small

$$H = \nabla_X u = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ - & - & - \\ - & - & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

$$\varepsilon = \max\left(\left| \frac{\partial u_i}{\partial x_j} \right| \right)$$



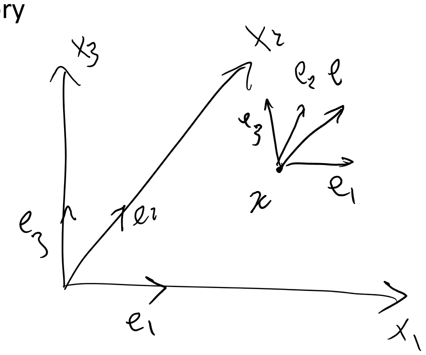
While this is not objective (different values in different coordinate system) it does the job for us to decide whether H is small or not

Infinitesimal theory

Infinitesimal theory

① $$\varepsilon = \max\left(\left|\frac{\partial u_i}{\partial x_j}\right|\right) \qquad \varepsilon \ll 1$$

What are the approximate values of normal and shear strains in infinitesimal theory



A) normal strain

$$\varepsilon(x,e) = \sqrt{e.Ce} - 1 = \mathcal{V}_e - 1$$

thus for e_i $(i=1,2,3)$

$$\varepsilon(e_i) = \sqrt{e_i . Ce_{\underline{i}}} - 1 \qquad \text{no summation}$$

$$= \sqrt{C_{i\underline{i}}} - 1$$

$$C = I + \overbrace{H + H^t}^{2G} + H^tH \qquad G = \frac{C-I}{2}$$

$$= I + 2G$$

$$\varepsilon(e_i) = \sqrt{C_{i\underline{i}}} - 1 = \sqrt{\underbrace{\delta_{ii}}_{1} + 2G_{i\underline{i}}} - 1 = \sqrt{1 + \underbrace{2G_{i\underline{i}}}_{=O(\varepsilon)}} - 1$$

very small #

$$G = \frac{H + H^t + H^tH}{2} = \underbrace{\frac{H + H^t}{2}}\ + \underbrace{\frac{H^tH}{2}} = O(\varepsilon)$$

if $H = O(\varepsilon)$

$$O(\varepsilon) + O(\varepsilon^2) = O(\varepsilon)$$

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we use the expansion of square root

$$\sqrt{1+x} = 1 + \frac{1}{2}x + \underbrace{\frac{1}{2}\left(1 - \frac{1}{2}\right)x^2 \dots} = 1 + \frac{1}{2}x + O(x^2)$$

$|x| \ll 1$

$$\varepsilon(e_i) = \sqrt{1 + 2\underbrace{G_{i\underline{i}}}} - 1 = \left(1 + \frac{1}{2}(2G_{i\underline{i}}) + O(\varepsilon^2)\right) - 1$$

$$\varepsilon(e_i) = \sqrt{1+2\underbrace{G_{ii}}_{x=O(\varepsilon)\text{ above}}} - 1 = \left(1 + \frac{1}{2}(2G_{ii}) + O(\varepsilon^2)\right) - 1$$

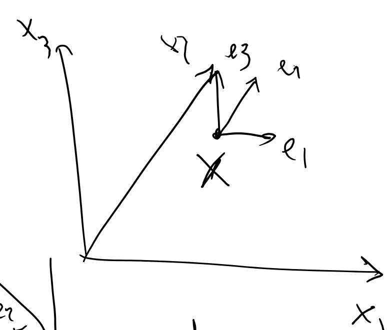
$$\sqrt{1+2G_{ii}}$$

$$\varepsilon(e_i) = G_{ii} + O(\varepsilon^2) = \overbrace{E_{ii}}^{(H+H^t)_{ii}/2} + \underbrace{\underbrace{(H^tH)_{ii}}_{H^t_{im}H_{mi}}}_{O(\varepsilon^2)} + O(\varepsilon^2)$$

(2)

$$\boxed{\begin{array}{l}\text{For normal strain} \\ \varepsilon(\cancel{X}e_i) = G_{ii} + O(\varepsilon^2) = E_{ii} + O(\varepsilon^2) \\ G = \frac{C-I}{2} = \underbrace{\left(\frac{H^t+H}{2}\right)}_{E} + \frac{H^tH}{2} \\ E = \frac{H^t+H}{2} \end{array}}$$

location

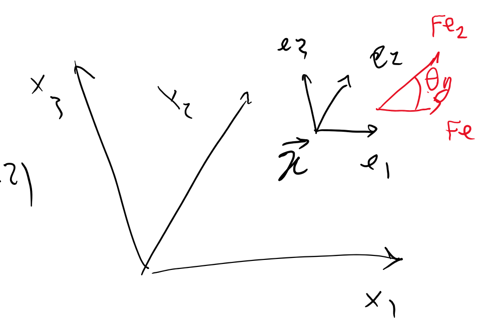


$$E = \begin{bmatrix} E_{11} & & \\ & E_{22} & \\ & & E_{33} \end{bmatrix} \rightarrow \text{normal strains}$$

B. Approximation of shear strain

$$\gamma_{12} = \frac{\pi}{2} - \theta_y$$

last time $\quad \sin\gamma_{ij} = \frac{e_i \cdot Ce_j}{\sqrt{e_i \cdot Ce_i}\sqrt{e_j \cdot Ce_j}}$ $\quad$ (eg $i=1, j=2$) $\quad i \neq j$



$$\underline{C = I + 2G} \qquad C_{ij} = \underbrace{\delta_{ij}}_{0} + 2G_{ij} \quad i \neq j \qquad C_{ij} = 2G_{ij} \quad i \neq j$$

$i \neq j$

$$\sin\gamma_{ij} = \frac{2G_{ij}}{\sqrt{1+2G_{ii}+O(\varepsilon^2)}\sqrt{1+2G_{jj}+O(\varepsilon^2)}} =$$

$$2\overbrace{G_{ij}}^{O(\varepsilon)}\left(1 - \frac{1}{2}\times 2(G_{ii}+O(\varepsilon^2))\right)\left(1 - \frac{1}{2}(2G_{jj}+O(\varepsilon^2))\right)$$

$$2G_{ij}\left(1-\frac{1}{2}\times 2\,\underbrace{\left(G_{ii}+O(\varepsilon^2)\right)}_{O(\varepsilon)}\right)\left(1-\underbrace{\frac{1}{2}\left(2G_{jj}+O(\varepsilon^2)\right)}_{O(\varepsilon)}\right)$$

$$\underbrace{\sin\gamma_{ij}}_{y} = \underbrace{\underbrace{2G_{ij}}_{O(\varepsilon)} + O(\varepsilon^2)}_{O(\varepsilon)} \quad \text{pretty small}$$

$$\sin y = y + \frac{y^3}{6} - \frac{y^5}{5!} \cdots$$

$$\sin y = y + O(y^3)$$

γ_{ij} will be $O(\varepsilon)$ similar to $\sin\gamma_{ij}$

$$\gamma_{ij} + O(\varepsilon^3) = 2G_{ij} + O(\varepsilon^2)$$

$$\Longrightarrow \quad \frac{\gamma_{ij}}{2} = G_{ij} + O(\varepsilon^2)$$

$$G_{ij} = E_{ij} + \underbrace{\left(\frac{H^t H}{2}\right)_{ij}}_{O(\varepsilon^2)}$$

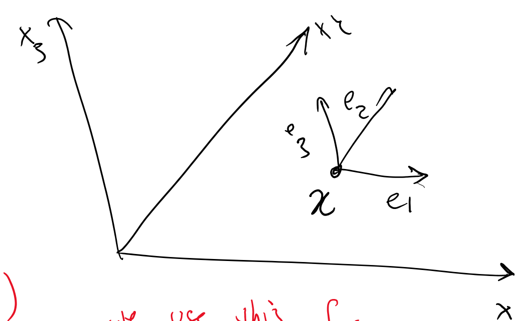
$$= \Big(\underbrace{E_{ij} + O(\varepsilon^2)}_{G_{ij}}\Big) + O(\varepsilon^2)$$

$$= E_{ij} + O(\varepsilon^2)$$

Summary of normal & shear strains

normal

$$\varepsilon(x, e_i) = \sqrt{e_i \cdot C e_i} - 1 = U e_i - 1 \quad \text{finite strain}$$



$$\varepsilon(x, e_i) = G_{ii} + O(\varepsilon^2) = E_{ii} + O(\varepsilon^2)$$

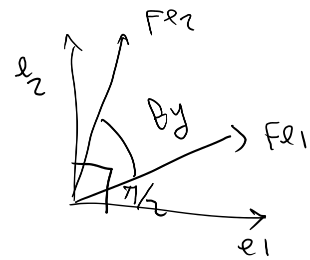
we use this for infinitesimal theory

shear ones

shear ones

$$\gamma_{ij} = \frac{e_i \cdot C e_j}{\sqrt{e_i \cdot C e_i}\,\sqrt{e_j \cdot C e_j}}$$ finite strain $i \neq j$

$$\epsilon_{ij}(x) = \frac{\gamma_{ij}(x)}{2} = G_{ij} + O(\varepsilon^2) = E_{ij} + O(\varepsilon^2)$$



$\gamma_{12} = \frac{\pi}{2} - \theta_y$ engineering shear strain

$\epsilon_{12} = \frac{\gamma_{12}}{2}$ (mathematical " ")

shear strains $= \frac{1}{2}\gamma_{ij}$

$$G = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ & G_{22} & G_{23} \\ \text{sym} & & G_{33} \end{bmatrix} \qquad E = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ & E_{22} & E_{23} \\ & & E_{33} \end{bmatrix}$$

normal strains

2nd order tensor rep of strain → "

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \end{bmatrix}$$ Voigt strain array

3

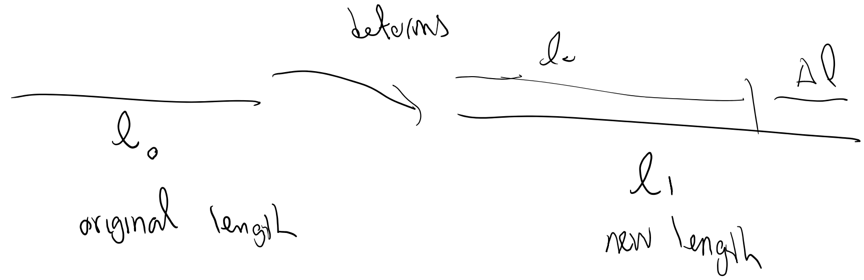
- E is an excellent strain measure for infinitesimal theory
- C, U, G are all great choices to represent deformation or strain in finite strain theory (no approximation). Therein, G is the strain measure that often is used.

4

$$G = \frac{C - I}{2}$$

More general definitions of strain

More general definitions of strain
Motivation:
1D



$$\varepsilon = \frac{\Delta l}{l_0} = \frac{l_1 - l_0}{l_0}$$

$\Delta l = l_1 - l_0$

How about this

intermediate stage

two choices

$0 \le dl \le \Delta l$

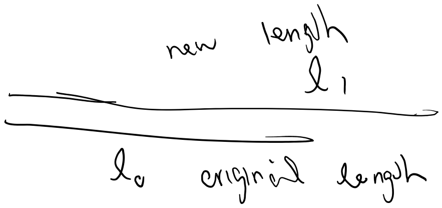
$l_0 \le l \le l_1$

$$\underbrace{d\varepsilon}_{\text{change in strain}} = \frac{dl}{l} \quad \text{(a)} \qquad \frac{dl}{l_0} \quad \text{(b)}$$

choice (a) $\quad \varepsilon = \int d\varepsilon = \int_{l_0}^{l_1} \frac{dl}{l} = \ln l \Big|_{l_0}^{l_1} = \ln\left(\frac{l_1}{l_0}\right)$

Logarithmic strain

choice (b) $\quad \varepsilon = \int d\varepsilon = \int_{l_0}^{l_1} \frac{dl}{l_0} = \frac{l_1 - l_0}{l_0}$ $\quad$ choice we first discussed



$\Delta l = l_1 - l_0$

(4) $\quad \varepsilon = \frac{\Delta l}{l_0}$ $\quad$ linear strain

$$\left(d\varepsilon = \frac{dl}{l_0}\right)$$

$$\varepsilon = \ln\left(\frac{l_1}{l_0}\right) = \ln\left(1 + \frac{\Delta l}{l_0}\right) = \frac{\Delta l}{l_0} + O\left(\left(\frac{\Delta l}{l_0}\right)^2\right)$$ logarithmic

$$\varepsilon = \ln\left(\frac{l_1}{l_0}\right) = \ln\left(1 + \frac{\Delta l}{l_0}\right) = \frac{\Delta l}{l_0} + O\left(\left(\frac{\Delta l}{l_0}\right)^2\right)$$ logarithmic

for this one we used $d\varepsilon = \frac{dl}{l}$

Example

original l_0 — new length $\frac{1}{2} l_0$

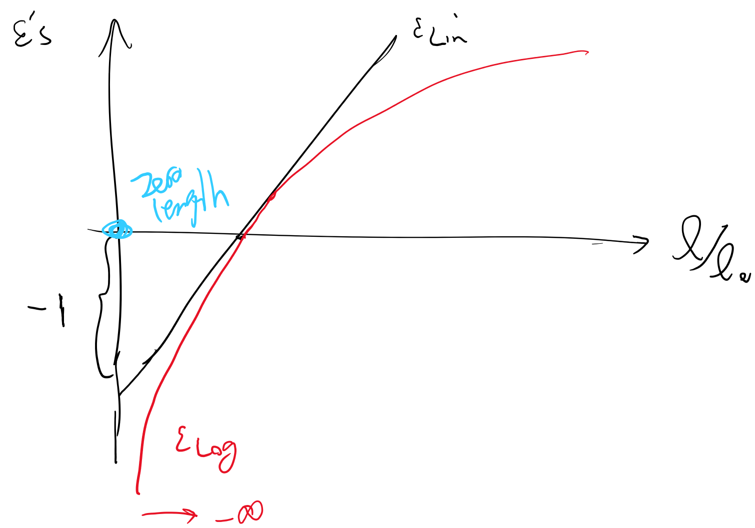
$$\varepsilon_{Lin} = \frac{\frac{1}{2} l_0 - l_0}{l_0} = \frac{-1}{2}$$

$$\varepsilon_{Log} = \ln\left(\frac{\frac{1}{2} l_0}{l_0}\right) = \ln\left(\frac{1}{2}\right)$$

$l_f = 0.001\, l_0$

$$\varepsilon_{Lin} = \frac{0.001 l_0 - l_0}{l_0} = -.999$$

$$\varepsilon_{Log} = \ln\left(\frac{0.001 l_0}{l_0}\right)$$



what if we want to have stress value

$$\sigma = E\varepsilon$$

As we can see a logarithmic strain can result in simpler constitutive models in certain cases (e.g. high compressive loading scenarios)

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strain = stretch -1

$$\varepsilon_i = \sqrt{C_{\underline{ii}}} - 1 = U_{\underline{ii}} - 1$$

$$U - I \quad \text{exact strain}$$

$$\frac{1}{2}\left(U^2 - I\right) = \frac{1}{2}\left(C - I\right) = G$$

$$\ln U \quad \text{logarithmic strain (Hencky strain)}$$

$$\frac{1}{m}\left(U^m - I\right) \quad \text{generalized Green strain}$$

m: nonzero integer ≥ 1