CM2023/11/06

Monday, November 6, 2023 9:38 AM

$U - I$ exact strain

$\frac{1}{2}(U^2 - I) = \frac{1}{2}(C - I) = G$

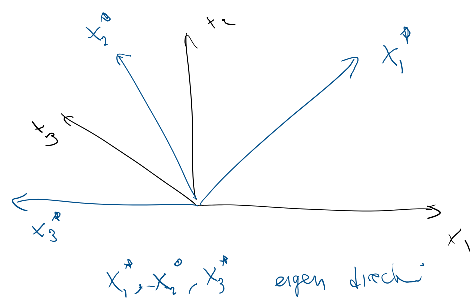
$\ln U$ logarithmic strain (Hencky strain)

$\frac{1}{m}(U^m - I)$ generalized Green strain

m: nonzero integer ≥ 1

How do we calculate these strain?

$C = F^t F$ pos def, sym →



x_1^*, x_2^*, x_3^* eigen direction

$$[C^*] = \begin{bmatrix} C_{11}^* > 0 & & \\ & C_{22}^* > 0 & \\ & & C_{33}^* > 0 \end{bmatrix}$$

$$U = \sqrt{C} \qquad [U^*] = \begin{bmatrix} \sqrt{C_{11}^*} & & \\ & \sqrt{C_{22}^*} & \\ & & \sqrt{C_{33}^*} \end{bmatrix}$$

How to calculate these strains?

$e(\lambda)$ general strain def.

(λ: eigen values of U)

exact linear strain

1. $e(\lambda) = \lambda - 1$

$$\begin{bmatrix} U_{11}^* & & \\ & U_{22}^* & \\ & & U_{33}^* \end{bmatrix} - I \quad \text{: strain is } U - I$$

1D $\frac{\ell}{\ell_0}$ $\frac{\Delta \ell}{\ell_0}$ $d\varepsilon = \frac{d\ell}{\ell_0}$

2. $e(\lambda) = \ln \lambda$

$$\begin{bmatrix} \ln U_{11}^* & & \\ & \ln U_{22}^* & \\ & & \ln U_{33}^* \end{bmatrix}$$

$\ln U$

1D $d\varepsilon = \frac{d\ell}{\ell} \rightarrow \varepsilon = \ln(\frac{\ell}{\ell_0})$

3) $e(\lambda) = \frac{1}{2}(\lambda^2 - 1)$

$$\begin{bmatrix} \frac{U_{11}^{*2} - 1}{2} & & \\ & \frac{(U_{22}^*)^2 - 1}{2} & \\ & & \frac{(U_{33}^*)^2 - 1}{2} \end{bmatrix}$$

$G = \frac{U^2 - I}{2} = \frac{C - I}{2}$ Green St Venant strain

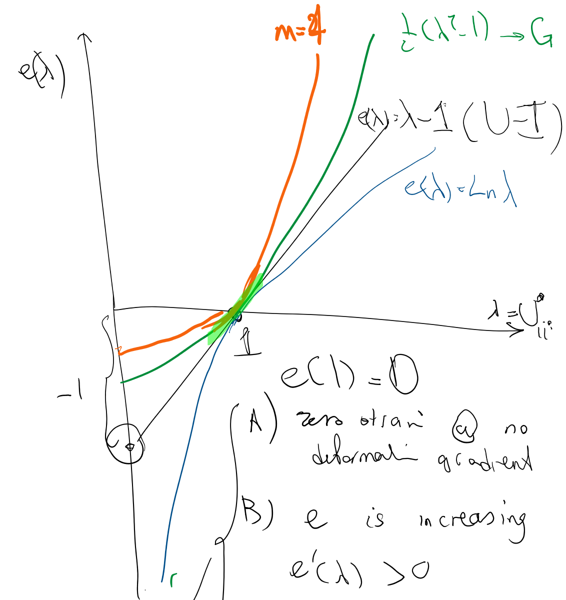
easy to calculate (C rather than $U = \sqrt{C}$ is used)

popular choice for large deformation formulation

4) $e(\lambda) = \frac{1}{m}(\lambda^m - 1)$

generalized green strain

$e(\lambda)$ ↑ m=4 | / $\frac{1}{2}(\lambda^2 - 1) \rightarrow G$



$e(1) = 0$

A) zero strain @ no deformation gradient

B) e is increasing
$e'(\lambda) > 0$

C) $e'(\lambda=1) = 1$

all these strains are almost equal for infinitesimal theory

$$e(U) = E + \underline{O(\varepsilon^2)} \qquad \varepsilon = \|H\|$$

Going back to rigid motion

$$y = Qx + c \Rightarrow F = \nabla_x y = Q$$

rotation

$y = Qx + c \Rightarrow F = \nabla_x y = Q$

$F = RU = VR$ in general

for rigid motion: $R = Q$, $U = I$, $V = I$, $R = Q$

$U = I,\ V = I,\ C = I,\ B = I$

$G = \frac{C - I}{2} = 0$

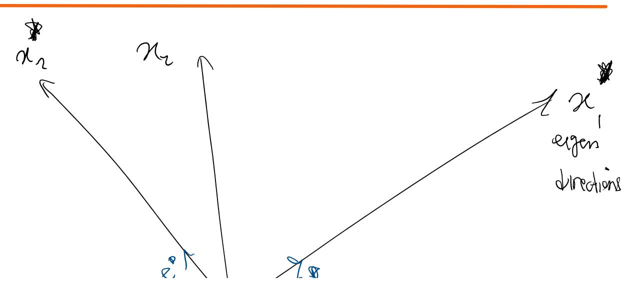
Summary

Finite deformation theory $\varepsilon = O(H) \not\ll 1$	Infinitesimal theory $\varepsilon = O(H) \ll 1$
$F = R U = V R$ U: Lagrangian, V: Eulerian R: rotation U, V = left, right stretches	$H = \nabla_x u = E + W$ $E = \frac{H + H^t}{2}$ infinitesimal strain $W = \frac{H - H^t}{2}$ 〃 rotation
side note plasticity → $F = F^e F^p$ F^e: elastic, F^p: plastic	$\varepsilon = \varepsilon_e + \varepsilon_p$
appropriate strains U, $G = \frac{U^2 - I}{2}$ ($C = U^2$), other $e(\lambda)$	E
Rigid motion $y = Qx + c$ $u = y - x = (Q - I)x + c$ approximated by → $F = Q$, $C = F^t F = I$ $G = 0$, $e(U) = 0$ zero strain $E = O(\varepsilon^2)$	$u = Wx + c$ W: skew $W = \frac{E - E^t}{2}$ $E = 0$ $G = O(\varepsilon^2)$

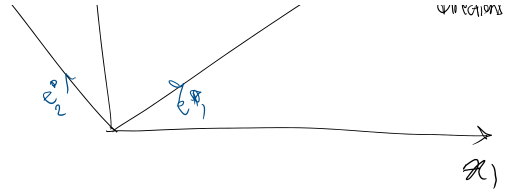
Mohr circle
Coordinate transformation for symmetric 2nd order tensors

$$E = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \qquad E_{12} = E_{21} = \frac{u_{1,2} + u_{2,1}}{2}$$

... the eigen problem $\det(E - \lambda I) = 0$



$$\ldots = \begin{bmatrix} \ldots & \ldots \\ E_{21} & E_{22} \end{bmatrix}$$



solve the eigen problem $\det(E - \lambda I) = 0$

$(E - \lambda I)V = \vec{0}$

V = eigen vector

$$[E^*] = \begin{bmatrix} E_{11}^* & \\ & E_{22}^* \end{bmatrix}$$ diagonal E in principal directions

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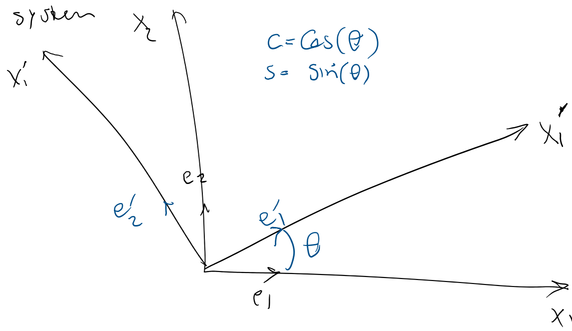
How to express E in another coordinate system

(e_1', e_2') may or may not be aligned with (e_1^*, e_2^*) above



$$\underbrace{[E']}_{\text{2nd order tensor}} = \begin{bmatrix} E_{11}' & E_{12}' \\ E_{21}' & E_{22}' \end{bmatrix}$$

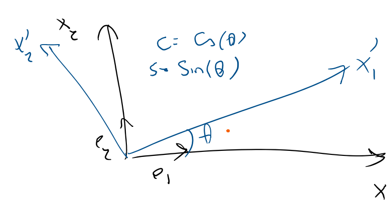
$$E_{ij}' = Q_{im} Q_{jn} E_{mn} = Q_{im} E_{mn} (Q^t)_{nj} \rightarrow [E'] = Q[E]Q^t \qquad Q = \begin{bmatrix} e_1' \\ e_2' \end{bmatrix}$$

expressed in e_1, e_2 system

$$[E'] = \underbrace{\begin{bmatrix} c & s \\ -s & c \end{bmatrix}}_{Q} \begin{bmatrix} E_{11} & E_{12} \\ \underbrace{E_{12}}_{\substack{E_{21} \\ \text{sym}}} & E_{22} \end{bmatrix} \underbrace{\begin{bmatrix} c & -s \\ s & c \end{bmatrix}}_{Q^t} \qquad Q = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

(1)

$$\boxed{\begin{aligned} E_{11}' &= c^2 E_{11} + s^2 E_{22} + 2cs\, E_{12} \\ E_{22}' &= s^2 E_{11} + c^2 E_{22} - 2cs\, E_{12} \\ E_{21}' = E_{12}' &= -cs(E_{11} - E_{22}) + (c^2 - s^2) E_{12} \end{aligned}}$$



note
$$c^2 = \cos^2\theta = \frac{\cos 2\theta + 1}{2}$$
$$s^2 = \sin^2\theta = \frac{1 - \cos 2\theta}{2}$$
$$cs = \frac{\sin 2\theta}{2}$$

$$C = \cos(2\theta)$$
$$S = \sin(2\theta)$$

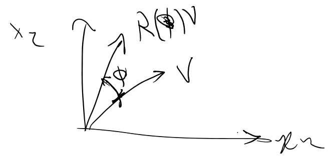
use these relations in (1) to get

$$\boxed{\begin{bmatrix} E_{11}' \\ E_{12}' \end{bmatrix} = \begin{bmatrix} \frac{E_{11} + E_{22}}{2} \\ 0 \end{bmatrix} + \underbrace{\begin{bmatrix} C & S \\ -S & C \end{bmatrix}}_{\text{rotation by } 2\theta} \begin{bmatrix} \frac{E_{11} - E_{22}}{2} \\ E_{12} \end{bmatrix}} \quad (2a)$$

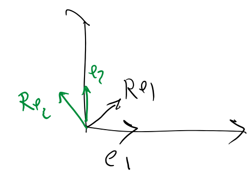
normal strain (E_{11}')
shear strain (E_{12}')

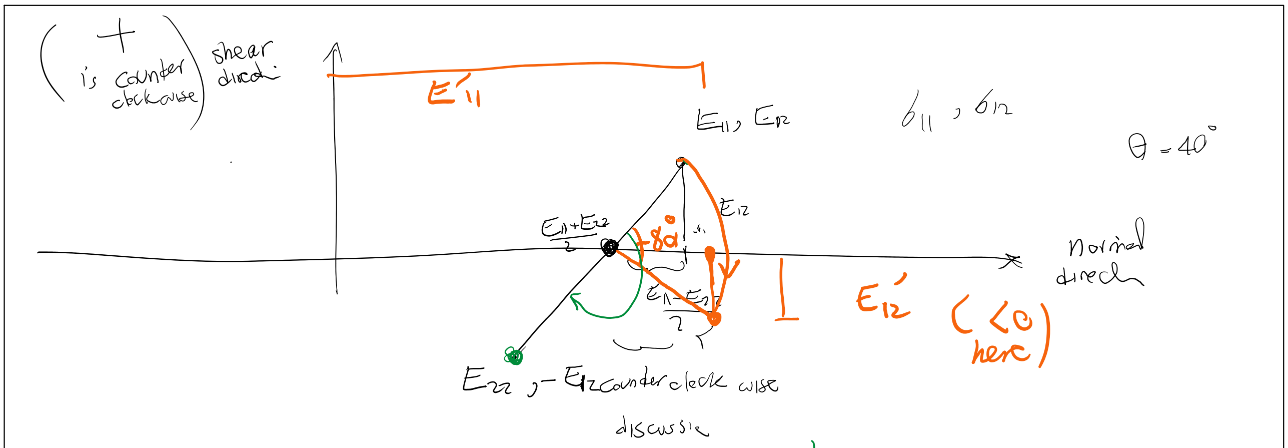
Recall Rotation by ϕ $\qquad R(\phi) = [\,Re_1 \mid Re_2\,]$

Recall Rotation by ϕ



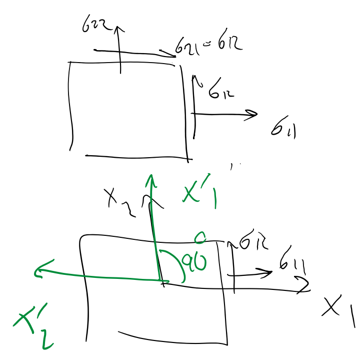
$$R(\phi) = \left[Re_1 \middle| Re_2 \right] = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$





(2b)

think of stress

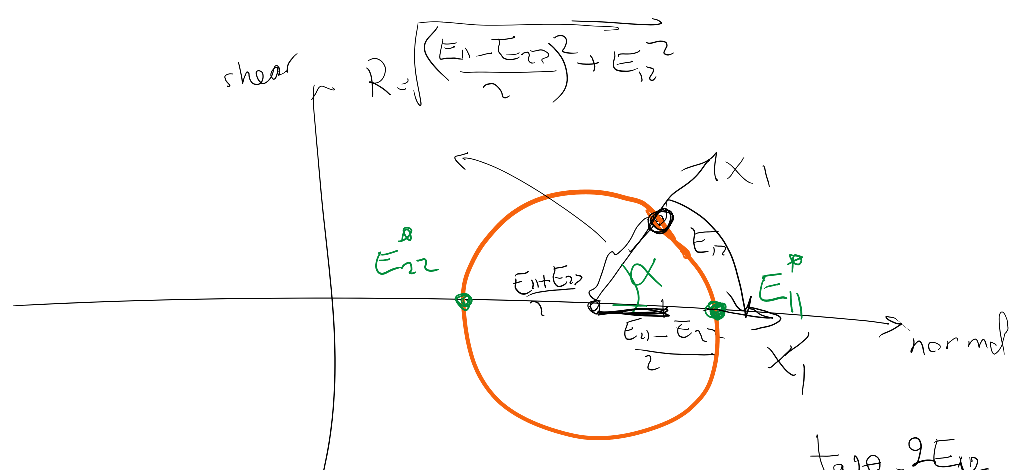


$$\begin{bmatrix} \frac{E_{11}+E_{22}}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} \cos(180°) & \sin(180°) \\ -\sin(180°) & \cos(180°) \end{bmatrix} \begin{bmatrix} \frac{E_{11}-E_{22}}{2} \\ E_{12} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{E_{11}+E_{22}}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{E_{11}-E_{22}}{2} \\ E_{12} \end{bmatrix} = \begin{bmatrix} E_{22} \\ -E_{12} \end{bmatrix}$$

$$R = \sqrt{\left(\frac{E_{11}-E_{22}}{2}\right)^2 + E_{12}^2}$$

$$E = \begin{bmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{bmatrix}$$

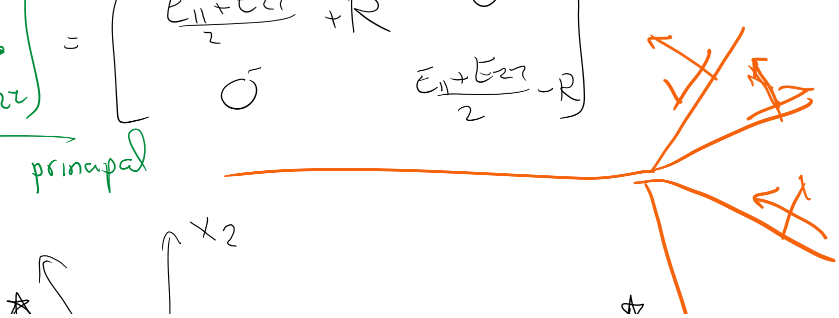


$$tg\,2\theta = \frac{2E_{12}}{E_{11}-E_{22}}$$

$$[E^*] = \begin{bmatrix} E_{11}^* & 0 \\ 0 & E_{22}^* \end{bmatrix} = \begin{bmatrix} \frac{E_{11}+E_{22}}{2}+R & 0 \\ 0 & \frac{E_{11}+E_{22}}{2}-R \end{bmatrix}$$

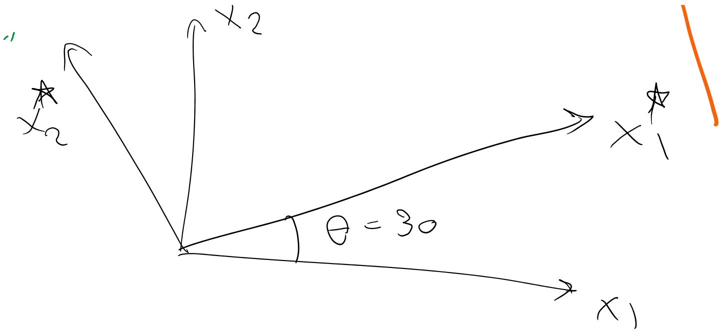
corresponding to principal directions

"principal values"



example use of this to determine crack growth direction

// principal values "
directions

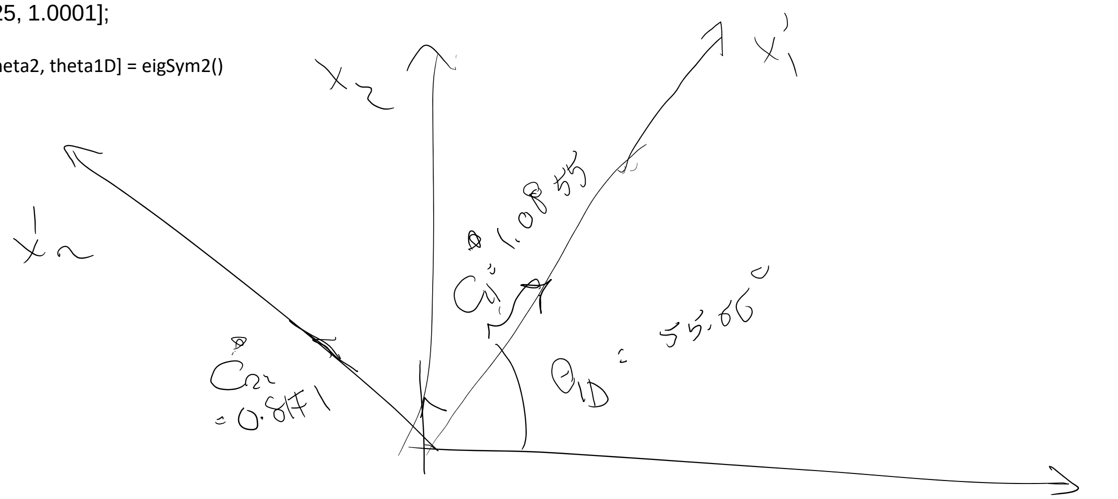


to determine crack growth direction

$x_1 \to x_1'$
$= -2\theta = -60^\circ$
$\theta = 30$

```
function [eigv, eigd, eigvalues, theta1, theta2,
theta1D] = eigSym2(C)
if nargin < 1
    C = [0.9025, 0.125;0.125, 1.0001];
```

[eigv, eigd, eigvalues, theta1, theta2, theta1D] = eigSym2()



eigv =

0.5641 -0.8257
0.8257 0.5641

eigd =

1.0855 0
0 0.8171

eigvalues =

1.0855 0.8171

$$[C^*] = \begin{bmatrix} 1.0855 & 0 \\ 0 & 0.8171 \end{bmatrix}$$

x_1

theta1 =

0.9715

theta2 =

2

Mohr circle in 3D

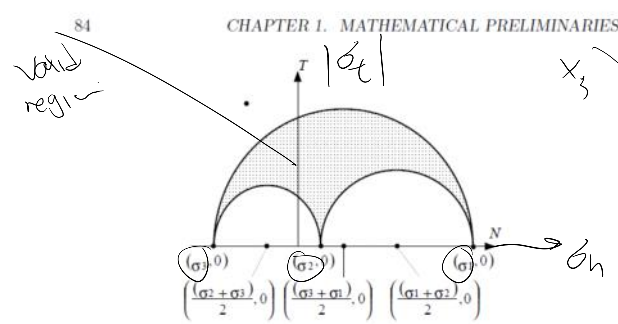
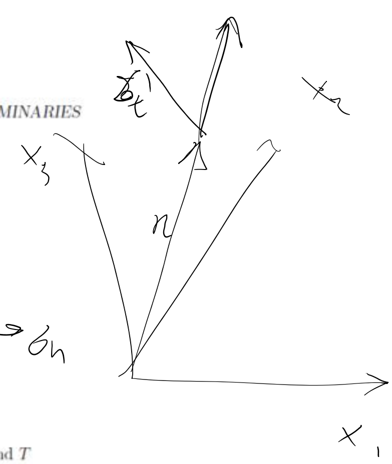


Figure 1.8: Mohr's Circle plot of extremal values for N and T



.542
3

theta1D =

55.6628