CM2023/11/08

Wednesday, November 8, 2023 9:43 AM

Definition 87 *A motion of a body is a family of deformations ordered by a single real parameter called* time, *denoted t. We introduce a reference time t_0 associated with the undeformed state of the body.*[16] *Then a motion is denoted by*

$$\{\mathbf{f}(\cdot,t)\}, t \in [t_0,\infty),$$

where

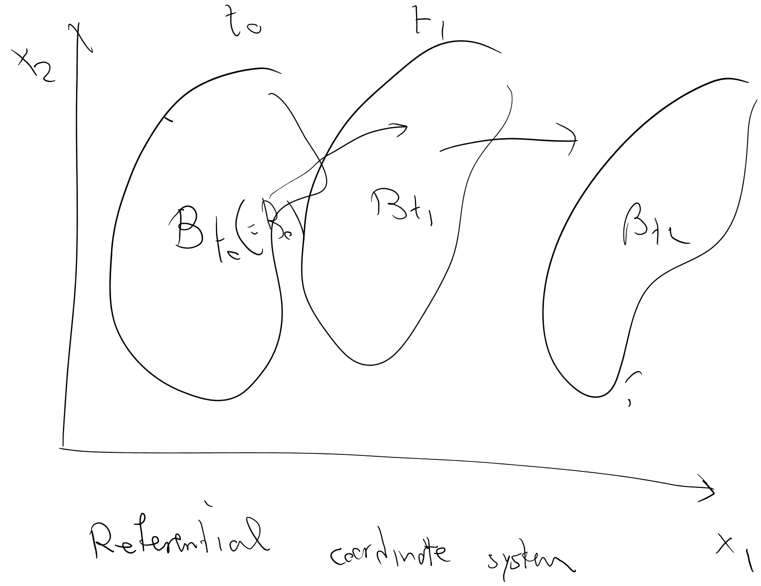
$$\mathbf{y} = \mathbf{f}(\mathbf{x},t)$$

is the position vector at time t of the material point identified by the position vector $\mathbf{x}$ in the undeformed state at time t_0. A motion inherits all the required properties of a deformation, except that the numbered properties in Definition 72 are superceded by the requirements

1. $\mathbf{f}(\mathbf{x},t_0) = \mathbf{x}$;
2. $\mathbf{f} \in C^2(\overset{0}{\mathcal{B}} \times [t_0,\infty), \mathcal{V})$.[17]

Why we don't have

det F > 0



Referential coordinate system

All prev sessions

$y = f(x)$

deformation

$y = f(x,t)$

Motion

Theorem 143 *Let $\{\mathbf{f}(\cdot,t)\}$ be a motion. Then*

$$J(\mathbf{x},t) > 0 \ \text{on}\ \overset{0}{\mathcal{B}} \times [t_0,\infty).$$

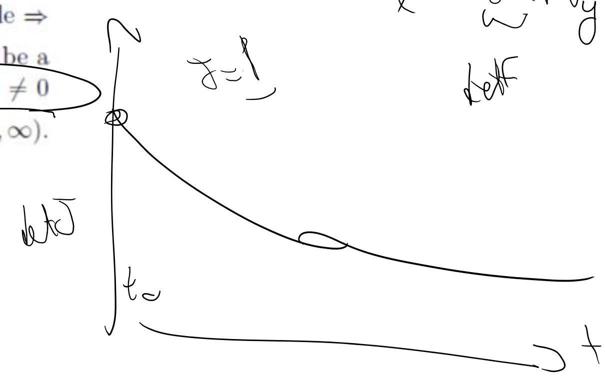
Proof. Our definition of a motion requires that for any fixed value of t, the mapping $\mathbf{f}(\cdot,t)$ is a deformation on $\overset{0}{\mathcal{B}}$. Therefore, $\mathbf{f}(\cdot,t)$ is invertible $\Rightarrow$ its Jacobian determinant $J(\mathbf{x},t) \neq 0$ on $\overset{0}{\mathcal{B}} \times [t_0,\infty)$. Since $J(\mathbf{x},t)$ must be a continuous function of both position and time, the requirement $J(\mathbf{x},t) \neq 0$ on $\overset{0}{\mathcal{B}} \times [t_0,\infty) \Rightarrow$ either $J(\mathbf{x},t) > 0$ or $J(\mathbf{x},t) < 0$ everywhere on $\overset{0}{\mathcal{B}} \times [t_0,\infty)$. At $t = t_0$ we have

$$\mathbf{f}(\mathbf{x},t_0) = \mathbf{x} \Rightarrow f_{i,j}(\mathbf{x},t_0) = \delta_{ij}.$$

Thus,

$$\begin{aligned} J(\mathbf{x},t_0) &= \det \mathbf{F}(\mathbf{x},t_0) \\ &= \det [f_{i,j}(\mathbf{x},t_0)] \\ &= \det [\delta_{ij}] \\ &= 1. \end{aligned}$$

$$dV_x = \underbrace{J}_{\det F} dV_y$$

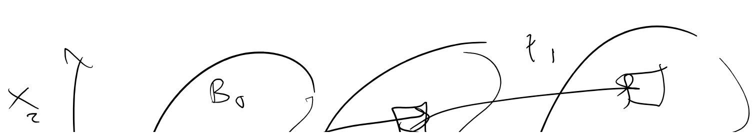


Definition of velocity:

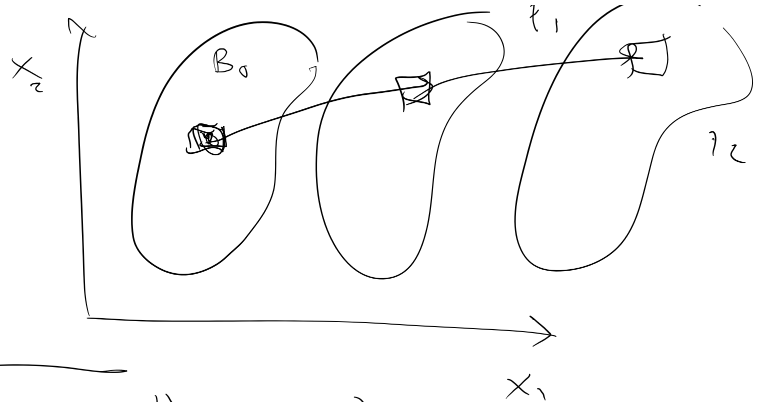
$$① \quad \boxed{v(x,t) = \frac{D\overbrace{f(x,t)}^{y}}{Dt}\Bigg|_{x\text{-fixed}}}$$

where

$$\boxed{\frac{D}{Dt} \ \text{or}\ \frac{\partial}{\partial t}\Bigg|_x \quad \text{material time derivative}} \quad ②$$



$$\frac{Dv(x,t)}{} \quad \frac{D^2 \overset{y}{f(x,t)}}{}$$



$$\boxed{a(x,t) = \frac{Dv(x,t)}{Dt} = \frac{D^2 \overset{y}{f}(x,t)}{Dt^2}}$$ (3)

$$u(x,t) = \underbrace{y(x,t)}_{\text{new position}} - \underset{\text{old (reference position)}}{x} \quad \therefore \quad \underset{\sim}{\overset{f}{y}}(x,t) = x + u(x,t)$$

$\frac{Dy}{Dt}\Big|_{x \text{ fixed}} =$

$$\cancel{v} = \frac{Du(x,t)}{Dt}\Big|_{x \text{ fixed}} + \frac{Dx}{Dt}\Big|_{\text{fixed}} = \frac{Du(x,t)}{Dt}$$

$y = f(x,t)$ shorthand is $y = y(x,t)$

similarly $\quad a = \dfrac{D^2 u(x,t)}{Dt^2}$

summary

$$\boxed{\begin{aligned} V(x,t) &= \frac{Du(x,t)}{Dt} = \frac{Dy(x,t)}{Dt} \\ a(x,t) &= \frac{D^2u(x,t)}{Dt^2} = \frac{D^2y(x,t)}{Dt^2} \end{aligned}}$$ (4)

$(X,t) \rightarrow$ Lagrangian (very common in solid mechanics)

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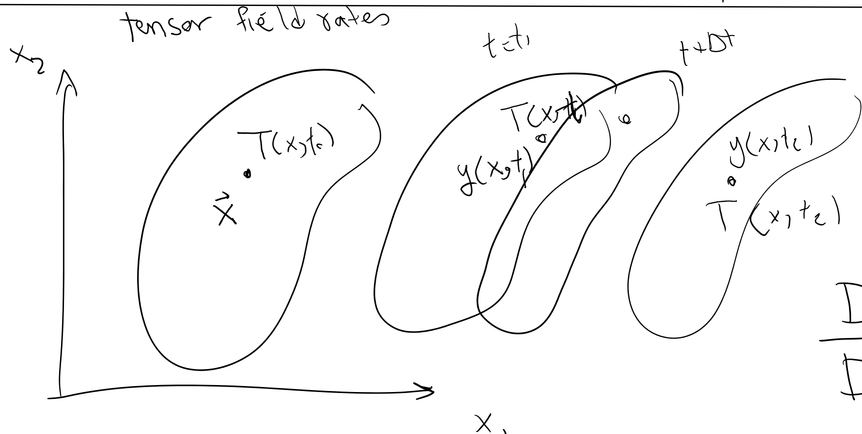
tensor field rates



Material rate of T
we follow the particle $\equiv$
T is written as a function of x

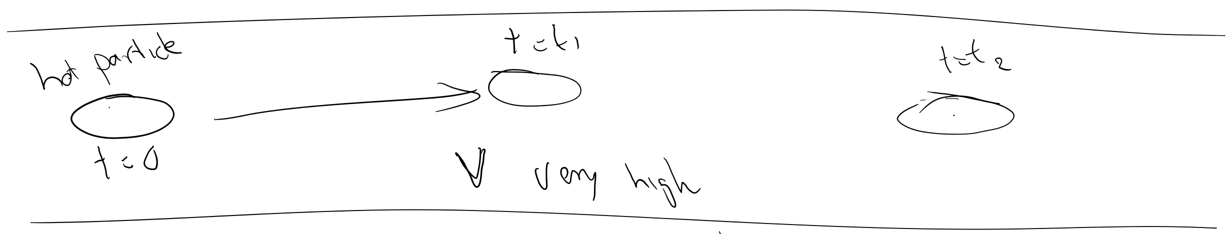
$$\frac{DT(x,t)}{Dt} = \lim_{\Delta t \to 0} \frac{T(x, t+\Delta t) - T(x,t)}{\Delta t}$$

(5) Lagrangian or referential time derivative

$$\frac{DT(x,t)}{Dt} = \lim \frac{T(x, t+\Delta t) - T(x,t)}{\Delta t} = \frac{\partial T}{\partial t}\Big|_{x\text{-fixed}}$$

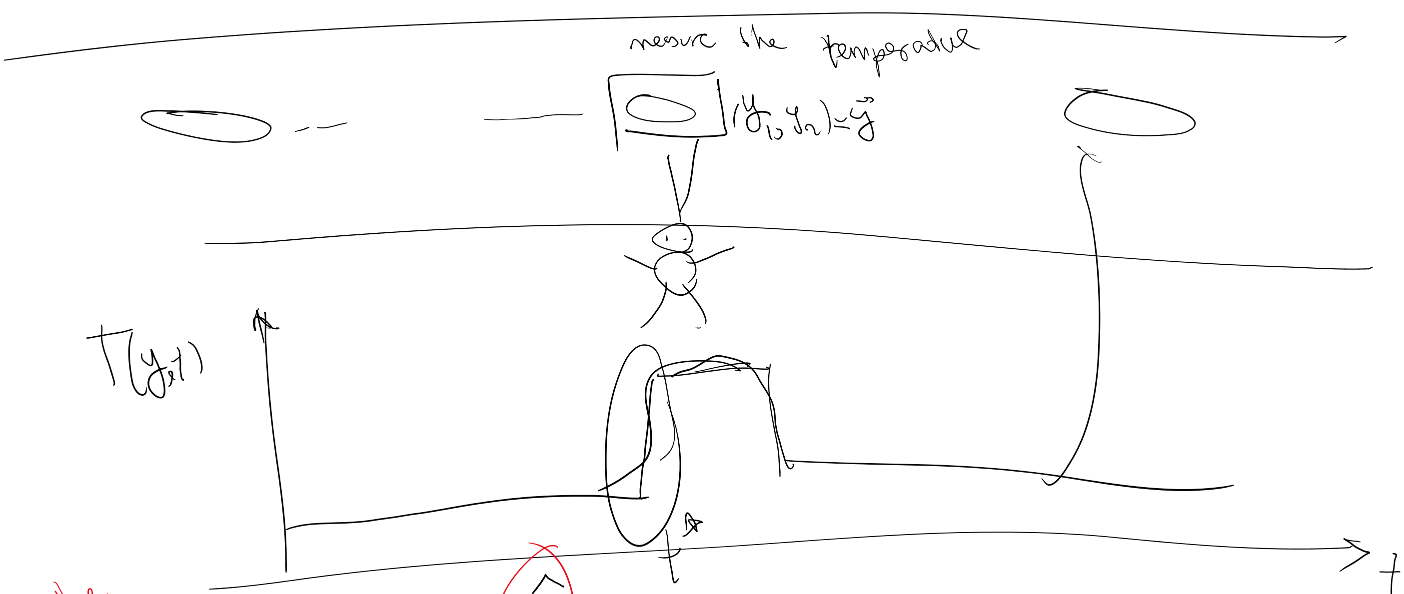
$$\boxed{\frac{DT(x,t)}{Dt} = \lim_{\Delta t \to 0} \frac{T(x, t+\Delta t) - \ldots}{\Delta t} = \frac{\partial T}{\partial t}\Big|_{x\text{-fixed}}}$$

Example



Because of conduction the temperature of the blub changes

$\frac{DT}{Dt}$ may be very small



the quantity is expressed in terms of y

$\frac{\partial \hat{T}(y,t)}{\partial t}$ at t^* is very high

Eulerian / Spatial time rate:

This is for a fixed position in space & we do not follow the material particle

Our goal is to relate the material rate (DT/Dt) (fixed x) to spatial rate (partial T/ partial t)

$$\left.\frac{D\hat{T}(y,t)}{Dt}\right|_{x\text{-fixed}} = \frac{\partial \hat{T}(y,t)}{\partial t} + \frac{\partial \hat{T}}{\partial y_i} \left.\underbrace{\frac{\partial y_i}{\partial t}}_{v_i}\right|_{x\text{-fixed}}$$

Chain rule

(6) $$\boxed{\underbrace{\left.\frac{D\hat{T}(y,t)}{Dt}\right|_{x\text{-fixed}}}_{\text{material/referential/Lagrangian rate}} = \underbrace{\frac{\partial \hat{T}(y,t)}{\partial t}}_{\text{spatial/Eulerian rate}} + \underbrace{\left(\nabla_y \hat{T}(y,t)\right)}_{\text{grad } \hat{T}} \hat{v}(y,t)}$$

grad w.r.t y

Example: $T = v$ velocity

(7)
$$\hat{a} = \frac{D\hat{v}}{Dt} = \frac{\partial \hat{v}(y,t)}{\partial t} + \left(\nabla_y \hat{v}(y,t)\right) \hat{v}(y,t)$$

shorter way of writing

$$a = \left.\frac{\partial v}{\partial t}\right|_{y\text{-fixed}} + (\nabla_y v)\, v$$

Conventions

	Material/Referential/Lagrangian	Spatial / Eulerian
	x	$y = \underset{\text{or } f}{\underset{\sim}{y}}(x,t)$
rate	$\frac{DT}{Dt}$ or $\left.\frac{\partial T}{\partial t}\right\|_x$	$\frac{\partial T}{\partial t}$ $\quad$ $\left.\frac{\partial T}{\partial t}\right\|_y$
Gradient	$\nabla_x T$ or Grad T	$\nabla_y T$ or grad T

(8)

Gradient	$\nabla_X T$ or Grad T	$\nabla_y T$ or grad T
Divergence	$\nabla_X \cdot T$ or Div T $=$ trace (Grad T)	$\nabla_y \cdot T$ or div T $=$ trace (grad T)

How are the grads related

Recall $F = \nabla_X y$

F_{kj}

Consider vector $w(X,t)$ $\quad \hat{w}(y,t)$

$$\left(\text{Grad } \hat{w}(y,t)\right)_{ij} = \frac{\partial \hat{w}_i(y,t)}{\partial X_j}\Big|_{X\text{-fixed}} = \frac{\partial \hat{w}_i(y,t)}{\partial y_k} \frac{\partial y_k}{\partial x_j}$$

$$= \underbrace{\frac{\partial \hat{w}_i(y,t)}{\partial y_k}}_{\text{grad } \hat{w}} F_{kj} \Longrightarrow \left(\text{Grad } \hat{w}\right)_{ij} = \left(\text{grad } \hat{w}\right)_{ik} F_{kj}$$

$$\boxed{\text{Grad } w = (\text{grad } w)\, F}$$

In fact, this relation holds for any order tensor:

F_{kj}

$$\left(\text{Grad } T(y,t)\right)_{i_1 \dots i_m j} = \frac{\partial \hat{T}_{i_1 \dots i_m}(y,t)}{\partial X_j} = \underbrace{\frac{\partial T_{i_1 \dots i_m}(y,t)}{\partial y_k}}_{(\text{grad } T)_{i_1 \dots i_m k}} \frac{\partial y_k}{\partial x_j}$$

T: m^{th} order tensor

$$= (\text{grad } T)_{i_1 \dots i_m k} F_{kj}$$

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$$\boxed{\text{For any tensor we have} \quad \begin{aligned} \text{Grad } T &= (\text{grad } T)\, F \\ \text{grad } T &= (\text{Grad } T)\, F^{-1} \end{aligned}}$$

We want to show the following,

$$\frac{\mathrm{D}\mathbf{T}}{\mathrm{D}t} = \frac{\partial \hat{\mathbf{T}}}{\partial t} + \mathrm{grad}\hat{\mathbf{T}}\hat{\mathbf{v}} \qquad \left.\frac{\mathrm{D}T_{i_1\cdots i_n}}{\mathrm{D}t}\right|_{\mathbf{x}} = \left.\frac{\partial \hat{T}_{i_1\cdots i_n}}{\partial t}\right|_{\mathbf{y}} + \frac{\partial \hat{T}_{i_1\cdots i_n}}{\partial y_k}\hat{v}_k \tag{2a}$$

$$\boxed{\mathrm{Grad}\mathbf{T} = \mathrm{grad}\hat{\mathbf{T}}.\mathbf{F}} \qquad \frac{\partial T_{i_1\cdots i_n}}{\partial x_j} = \frac{\partial \hat{T}_{i_1\cdots i_n}}{\partial y_k}F_{kj} \tag{2b}$$

for you in HW8

$$\mathrm{Div}\mathbf{T} = J\mathrm{div}(\mathbf{T}\mathbf{F}^{\mathrm{T}}/J) \qquad \frac{\partial T_{i_1\cdots i_n}}{\partial x_{i_n}} = J\frac{\partial(\hat{T}_{i_1\cdots i_n}F_{ji_n}/J)}{\partial y_j} \tag{2c}$$

Note that Div and Grad equations can also be written as,

$$\mathrm{grad}\hat{\mathbf{T}} = \mathrm{Grad}\mathbf{T}.\mathbf{F}^{-1} \qquad \frac{\partial \hat{T}_{i_1\cdots i_n}}{\partial y_k} = \frac{\partial T_{i_1\cdots i_n}}{\partial x_j}F^{-1}_{jk} \tag{3a}$$

$$\mathrm{Div}(J\mathbf{T}\mathbf{F}^{-\mathrm{T}}) = J\mathrm{div}\hat{\mathbf{T}} \qquad \frac{\partial J T_{i_1\cdots i_n}F^{-1}_{ji_n}}{\partial x_j} = J\frac{\partial \hat{T}_{i_1\cdots i_n}}{\partial y_{i_n}} \tag{3b}$$

Useful identity (for balance laws)

$$\underbrace{\frac{DJ}{Dt}}_{\text{material rate}} = ? \qquad J = \det F$$

$$\frac{DJ}{Dt} = \frac{D\det F}{Dt}$$

From the equation sheet:

$$\frac{\mathrm{d}(\det \mathbf{A})}{\mathrm{d}\alpha} = \mathrm{trace}\left(\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}\alpha}\mathbf{A}^{-1}\right)\det \mathbf{A}$$ α any argument (dependency) of $\mathbf{A}$ such as time t

$$\frac{D\det F}{Dt} = \mathrm{trace}\left(\underbrace{\frac{dF}{dt}}F^{-1}\right)\underbrace{\det F}_{J} \qquad \text{fixed } x \qquad (i)$$

$$\frac{DF}{Dt} = ? \qquad \text{fixed } x$$

$$\left(\frac{DF}{Dt}\right)_{ij} = \frac{\partial}{\partial t}\Bigg(\underbrace{\frac{\partial y_i}{\partial x_j}}_{F_{ij}}\Bigg)\Bigg|_{\text{fixed } x} = \frac{\partial}{\partial x_j}\underbrace{\left(\frac{\partial y_i}{\partial t}\right)\Bigg|_{x\text{-fixed}}}_{\text{velocity } (V_i)}$$

$$= \frac{\partial \dot{V}_i}{\partial X_j} \Big| \text{fixed } x \Rightarrow \boxed{\left(\frac{DF}{Dt}\right)_{ij} = (\text{Grad}V)_{ij} \quad \text{or} \quad \frac{DF}{Dt} = \text{Grad}V}$$

velocity (V_i)

plug $\dot{B}_i$ into $\underline{i}$

$$\frac{DJ}{Dt} = \text{trace}\Big(\underbrace{(\text{Grad}V)F^{-1}}_{\text{grad}V}\Big) J$$

Recall $\text{Grad}\,T = (\text{grad}\,T)F$
$\text{grad}\,T = (\text{Grad}\,T)F^{-1}$
for a tensor T
eqn 9

$$\frac{DF}{Dt} = \text{trace}(\text{grad } v) J$$

$$= (\text{div } v) J$$

Summary of all equations

Lagrangian

$$V = \frac{Dy(x,t)}{Dt} = \frac{Du(x,t)}{Dt}$$

$$a = \frac{Dv}{Dt} = \frac{D^2y}{Dt^2} = \frac{D^2u}{Dt^2}$$

Eulerian

$$\hat{v}(y,t)$$

$$\hat{a}(y,t) = \frac{\partial \hat{v}(y,t)}{\partial t} + (\text{grad}\,\hat{v})\,\hat{v}$$

in general

$$\frac{D\hat{T}}{Dt} = \frac{\partial \hat{T}}{\partial t} + (\text{grad}\,T)\,\hat{v}$$

$$\text{Grad}\,T = \text{grad}\,T\,F$$

$$\frac{DJ}{Dt} = J(\text{div}\,v)$$

Notation

sidenote

$$\frac{DF}{Dt} = \text{Grad}\,V = \underbrace{(\text{grad}\,v)}_{L}\,F$$

Spatial gradient of velocity

$$D = \frac{L + L^t}{2}$$ Stretching

$$E = \frac{H + H^t}{2}$$

Compare with E in solid mechanics

$$W = \frac{L - L^t}{2}$$ spin

similar to solid $W = \frac{H - H^t}{2}$

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Abeyaratne_Continuum Mechanics_RCA_Vol_II.pdf

The *velocity gradient tensor* is defined as the (spatial) gradient of the velocity field:

$$\boldsymbol{L}(\boldsymbol{y},t) = \text{grad}\,\boldsymbol{v}(\boldsymbol{y},t) = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{y}}(\mathbf{y},t) \tag{3.16}$$

where we have now omitted the "overline" on $\mathbf{v}$. In terms of components in an orthonormal basis,

$$L_{ij} = \frac{\partial v_i}{\partial y_j}. \tag{3.17}$$

Note the useful fact that

$$\text{tr}\,\mathbf{L} = \text{div}\,\mathbf{v}. \tag{3.18}$$

On using (3.4), $(3.3)_1$ and (1.17) we find

$$\dot{\boldsymbol{F}} = \frac{\partial}{\partial t}(\text{Grad}\,\boldsymbol{y}(\boldsymbol{x},t)) = \text{Grad}\left(\frac{\partial \boldsymbol{y}}{\partial t}(\boldsymbol{x},t)\right) = \text{Grad}\,\boldsymbol{v} = (\text{grad}\,\boldsymbol{v})\boldsymbol{F} = \boldsymbol{L}\boldsymbol{F}. \tag{3.19}$$

This leads to the following useful expression for $\mathbf{L}$:

$$\boldsymbol{L} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}. \tag{3.20}$$

The *stretching tensor* (or *rate of deformation tensor*) $\boldsymbol{D}$ and the *spin tensor* $\boldsymbol{W}$ are defined as the symmetric and skew-symmetric parts of $\boldsymbol{L}$:

$$\boldsymbol{D} = \tfrac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^T), \qquad \boldsymbol{W} = \tfrac{1}{2}(\boldsymbol{L} - \boldsymbol{L}^T); \tag{3.21}$$

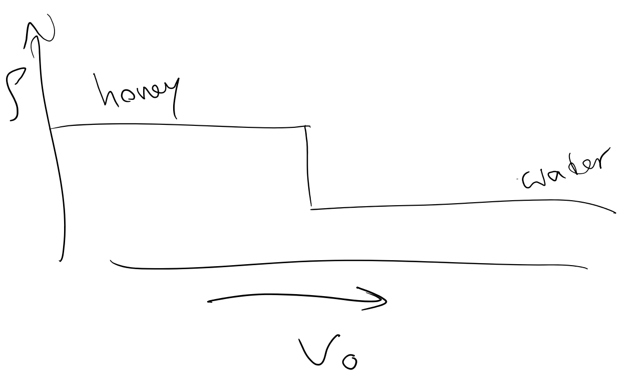
clearly the stretching tensor is symmetric, the spin tensor is skew-symmetric, and their sum equals the velocity gradient tensor:

$$\boldsymbol{D} = \boldsymbol{D}^T \qquad \boldsymbol{W} = -\boldsymbol{W}^T \qquad \text{and} \qquad \boldsymbol{L} = \boldsymbol{D} + \boldsymbol{W}. \tag{3.22}$$

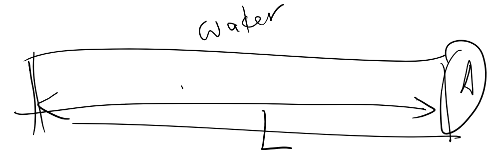
Balance of linear momentum

$$F = ma = \frac{d(mv)}{dt}$$ linear momentum

particle

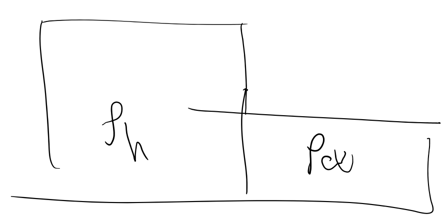


time t_0



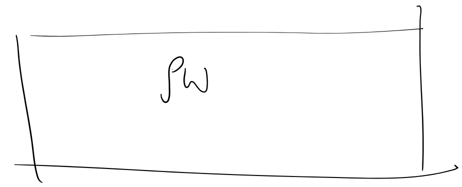
linear momentum $\leftarrow P_0 = MV = \underbrace{(\rho_w L A)}_{M} V_0$

time t_1



$$P_0 < P_1 = \left(\frac{\rho_w + \rho_h}{2}\right) L A V_0$$

time t_2



$$P_0 < P_1 < P_2 = (\rho_h A L)\, V_c$$

$$\frac{dP}{dt} > 0 \qquad F = \frac{dP}{dt} > 0$$

This is incorrect and will be explained further next time