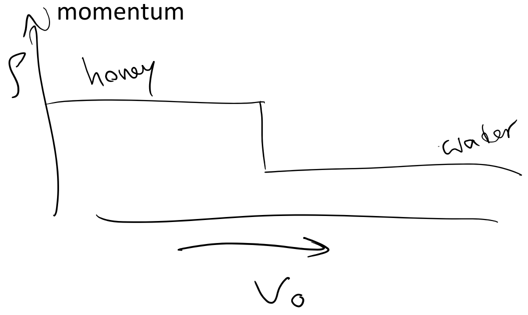
CM2023/11/13

Monday, November 13, 2023 9:38 AM

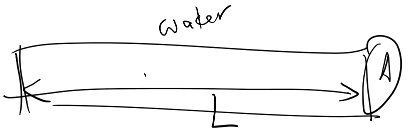
Balance of linear momentum



particle

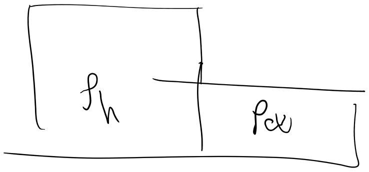
$$F = ma = \frac{d\,(mv)}{dt} \quad \text{linear momentum}$$

time t_0



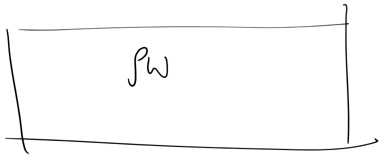
$$\text{linear momentum} \longleftarrow P_0 = MV = \underbrace{(\rho_w L A)}_{M} V_0$$

time t_1



$$P_0 < \quad P_1 = \left(\frac{\rho_w+\rho_h}{2}\right) L A V_0$$

time t_2



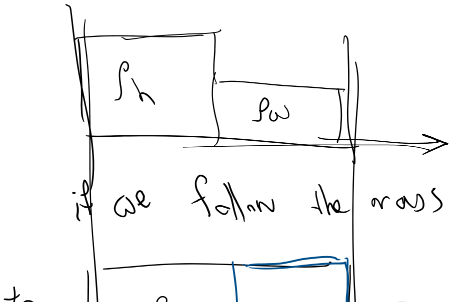
$$P_0 < P_1 < P_2 = (\rho_h A L)\, V_0$$

$$\frac{dP}{dt} > 0 \qquad F = \frac{dP}{dt} > 0$$

This is incorrect and will be explained further next time

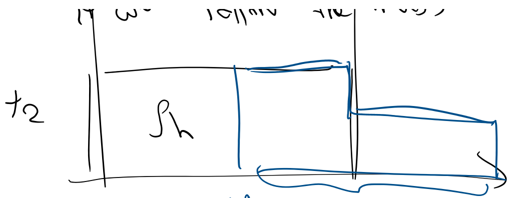
The problem is that we looked at fixed region in space (Eulerian viewpoint) whereas we should follow the mass

@ t_1

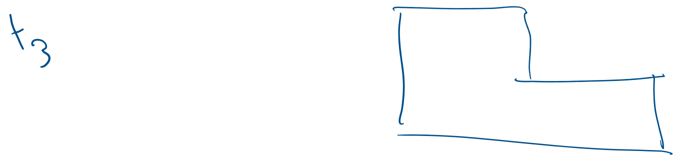


if we follow the mass

$$P_1 = \left[\left(\frac{\rho_h + \rho_w}{2}\right) L A\right] V_0$$



however if we consider the "same" material region $P_2 = \left(\frac{\rho_h+\rho_w}{2}\right)LAV_0$

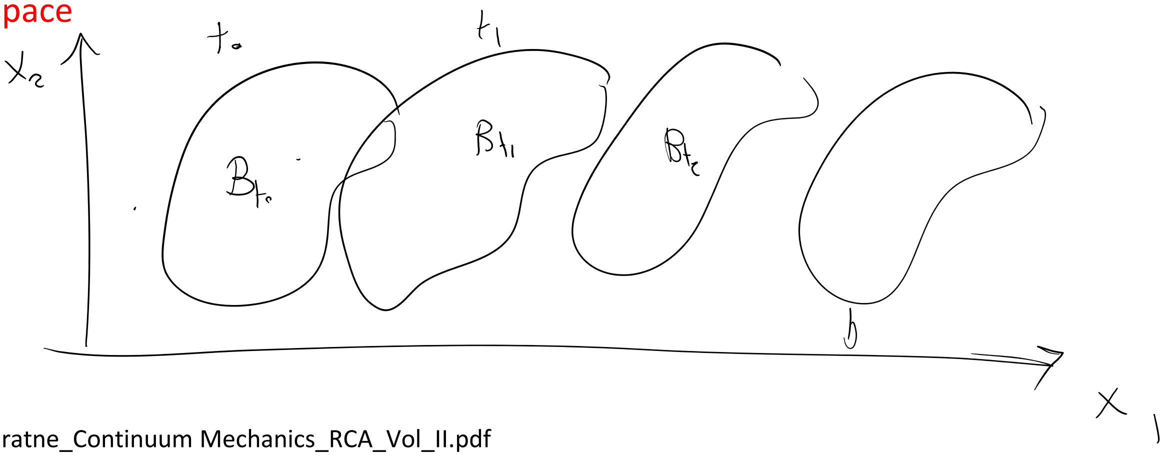


$P_3 = P_2$

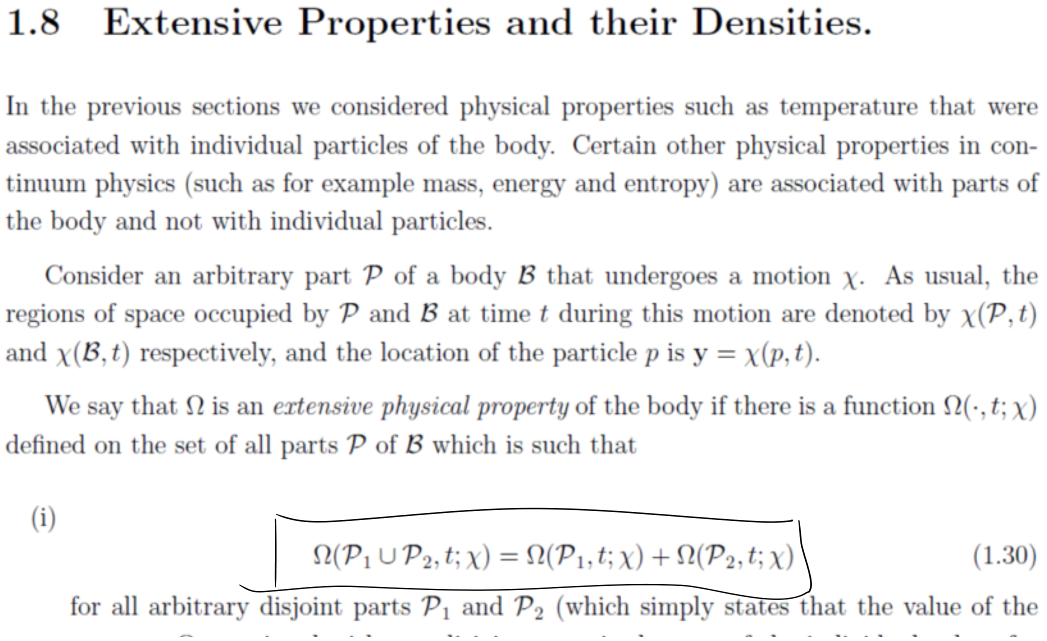
P stays constant $F = \boxed{\frac{DP}{D}} = 0$

material derivative

For balance laws, we need to follow the same region of material as it moves in space



Abeyaratne_Continuum Mechanics_RCA_Vol_II.pdf



1.8 Extensive Properties and their Densities.

In the previous sections we considered physical properties such as temperature that were associated with individual particles of the body. Certain other physical properties in continuum physics (such as for example mass, energy and entropy) are associated with parts of the body and not with individual particles.

Consider an arbitrary part $\mathcal{P}$ of a body $\mathcal{B}$ that undergoes a motion χ. As usual, the regions of space occupied by $\mathcal{P}$ and $\mathcal{B}$ at time t during this motion are denoted by $\chi(\mathcal{P},t)$ and $\chi(\mathcal{B},t)$ respectively, and the location of the particle p is $\mathbf{y} = \chi(p,t)$.

We say that Ω is an *extensive physical property* of the body if there is a function $\Omega(\cdot,t;\chi)$ defined on the set of all parts $\mathcal{P}$ of $\mathcal{B}$ which is such that

(i) $$\boxed{\Omega(\mathcal{P}_1 \cup \mathcal{P}_2, t; \chi) = \Omega(\mathcal{P}_1, t; \chi) + \Omega(\mathcal{P}_2, t; \chi)} \qquad (1.30)$$

for all arbitrary disjoint parts $\mathcal{P}_1$ and $\mathcal{P}_2$ (which simply states that the value of the

$$\Omega(\mathcal{P}_1 \cup \mathcal{P}_2, t; \chi) = \Omega(\mathcal{P}_1, t; \chi) + \Omega(\mathcal{P}_2, t; \chi) \qquad (1.30)$$

for all arbitrary disjoint parts $\mathcal{P}_1$ and $\mathcal{P}_2$ (which simply states that the value of the property Ω associated with two disjoint parts is the sum of the individual values for

property Ω associated with two disjoint parts is the sum of the individual values for

each of those parts), and

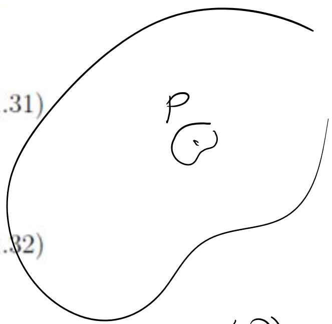
Ω = mass

(ii)

$$\Omega(\mathcal{P}, t; \chi) \to 0 \quad \text{as the volume of } \chi(\mathcal{P}, t) \to 0. \qquad (1.31)$$

Under these circumstance there exists a *density* $\omega(p, t; \chi)$ such that

$$\Omega(\mathcal{P}, t; \chi) = \int_{\mathcal{P}} \omega(p, t; \chi)\, dp. \qquad (1.32)$$



Mass

$$\rho(\cancel{x}) = \lim \frac{\text{Mass}(P)}{\text{Vol}(P)}$$
as $\text{Vol}(P) \to 0$

$$\omega = \lim \frac{\Omega(P)}{\|P\|}$$
as $\|P\| \to 0$

$$\underset{\text{Mass}}{M} = \int_{P} \underbrace{\rho}_{\text{Mass density}} dV$$

other examples

$$\underline{\text{linear momentum}}\ P = \sum m\vec{V} = \int_{P} \vec{V}\, dm$$

$$= \int \underset{\text{vel}}{\vec{V}} (\rho\, \underset{\text{vol}}{dV}) = \int_{P} \underbrace{(\rho\vec{V})}_{p = \rho\vec{V}} d\bar{V}$$

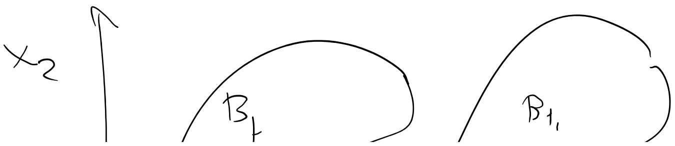
$p = \rho\vec{V}$ = linear momentum density

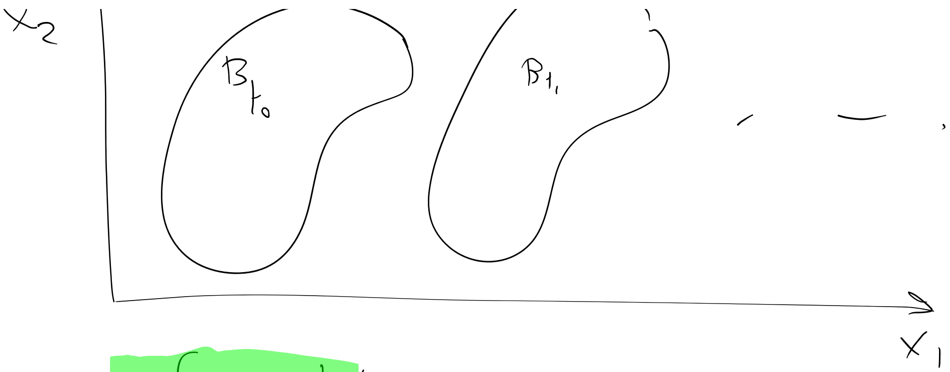
Energy

$$E = \int_{P} \underbrace{e}_{\text{volumetric energy density}} dV$$

volumetric energy density $\left(\frac{\text{Energy}}{\text{Volume}}\right)$

Balance laws look like this

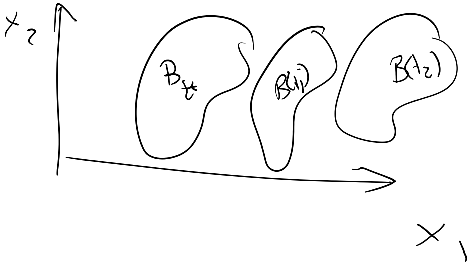




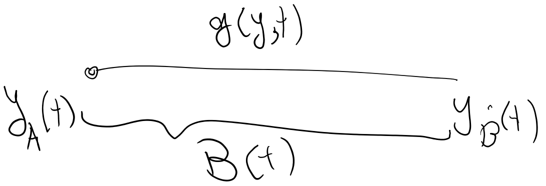
$$\underbrace{\frac{D}{Dt}}_{\text{material derivative}} G = \frac{D}{Dt}\int_B \overset{}{g}\, dV = \text{fluxes through the boundary "+" source terms inside}$$

$g \downarrow$ density of G

How do we calculate $\frac{D}{Dt}\int_{B(t)} g\,dV$ where $B(t)$ is moving in time?

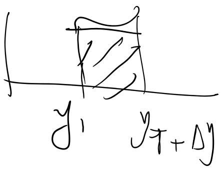


Example in 1D:



$$G(t) = \int_{y_A(t)}^{y_B(t)} g(y,t)\, dt$$

Want to calculate $\frac{DG(t)}{Dt} = ?$

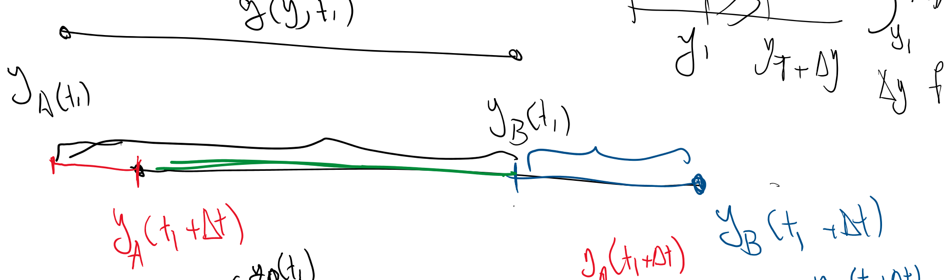


$\int_{y_1}^{y_1+\Delta y} f(y)\, dy$

$\Delta y\, f(y_1)$

@ $t = t_1$

@ $t = t_1+\Delta t$



$(y_B(t_1)$ $y_A(t_1+\Delta t)$ $y_R(t_1+\Delta t)$

$$\Delta G = G(t_2) - G(t_1) = \underbrace{\int_{y_A(t_1)}^{\overset{y_A(t_1+\Delta t)}{y_B(t_1)}} g(y,t_2)\,dt \longrightarrow \int_{y_A(t_1)}^{y_A(t_1+\Delta t)} g(y,t_2)\,dt \quad \overset{y_B(t_1+\Delta t)}{} + \int_{y_B(t_1)}^{y_B(t_1+\Delta t)} g(y,t_2)\,dt}_{G(t_2)}$$

$$- \underbrace{\int_{y_A(t_1)}^{y_B(t_1)} g(y,t_1)\,dt}_{G(t_1)}$$

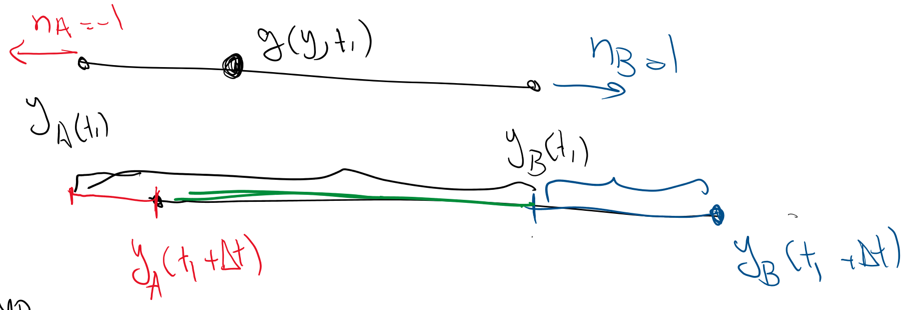
$$\approx \Delta t \left\{ \int_{y_A(t_1)}^{y_B(t_1)} \left(\frac{g(y,\overset{t_1+\Delta t}{\overset{\uparrow}{t_2}}) - g(y,t_1)}{\Delta t} \right) dt - \left(\left(\frac{y_A(t_1+\Delta t) - y_A(t_1)}{\Delta t} \right) \right) g(y_A,t_2) + \right.$$

$$\left. + \left(\frac{y_B(t_1+\Delta t) - y_B(t_1)}{\Delta t} \right) g(y_B,t_2) \right\}$$

↳ $\Delta t \longrightarrow 0$

$$\frac{G(t_1+\Delta t) - G(t_1)}{\Delta t} = \underbrace{\int_{y_A(t_1)}^{y_B(t_1)} \frac{\partial g(y,t)}{\partial t}\Big|_{y \text{ fixed}}}_{\text{Eulerian time derivative}} - \overbrace{\frac{d y_A}{dt}}^{V_A = \frac{dy_A}{dt}}(t_1)\, g(y_A,t_2)$$

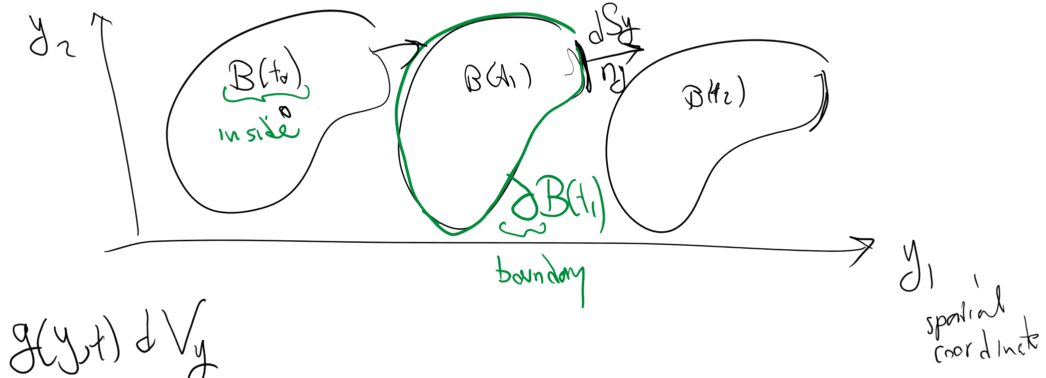
$$+ \underbrace{\frac{d y_B}{dt}}_{V_B = \frac{dy_B}{dt}}(t_1)\, g(y_B)$$



$$\frac{DG}{Dt} = \int_{y_A}^{y_B} \frac{\partial g(y,t)}{\partial t}\Big|_{y\text{-fixed}} dy + V_B\, g(y_B)\, n_B + V_A\, g(y_A)\, n_A$$

∂A $+ V_A\ g(y_A)\ n_A$

What about 2D,3D?

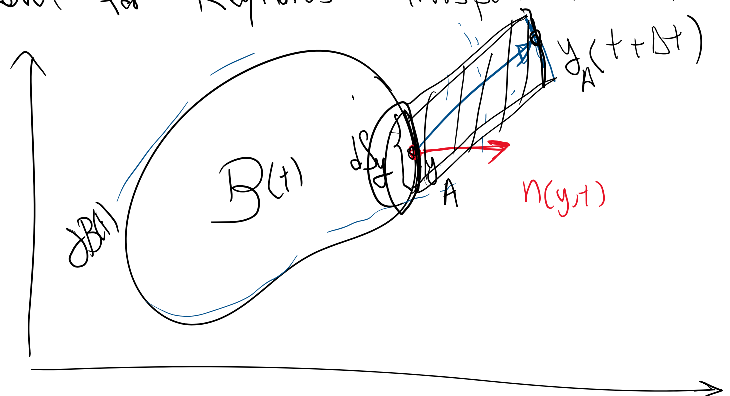


$$G(t) = \int_{B(t)} g(y,t)\, dV_y$$

$$\frac{DG}{Dt} = \underbrace{\int_{B(t)} \frac{\partial g(y,t)}{\partial t}\Big|_{y\text{ -fixed}} dV_y}_{\text{inside}} + \underbrace{\int_{\partial B(t)} g(y,t)\ V(y,t)\cdot\vec{n}_y\ \underbrace{dS_y}}_{\text{on the boundary}}$$

Reynold's transport theorem

Argument for Reynold's transport theorem



area in 2D

Added "volume"

$$= \left(dS_y\, \vec{n}_y\right)\cdot\left(y_A(t+\Delta t) - y_A(t)\right)$$

$$y_A(t+\Delta t) - y_A(t) = \frac{y_A(t+\Delta t) - y_A(t)}{\Delta t}\ \Delta t = V(y_A,t)\ \Delta t$$

as $\Delta t \to 0$

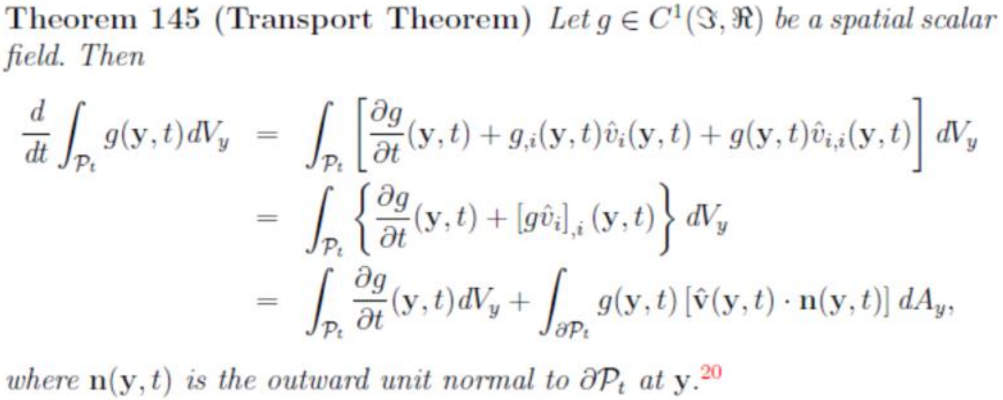
in material time derivative from t to $t+\Delta t$ we are adding

$$\underbrace{\underbrace{g(y_A,t)}_{\text{density of the quantity}} \Big(\underbrace{(dS_y\, n_y)\cdot V(y_A,t)\ \Delta t}_{\text{Added Volume}} \Big)}$$

$$\frac{DG}{Dt} = \underbrace{\text{change inside } B}_{} + \text{Contribution on the boundary}$$

$$= \int_{B(t)} \frac{\partial g(y,t)}{\partial t}\, dV_y + \int_{\partial B(t)} g(y,t)\, V_i(y,t) \cdot n(y,t)\, dS_y$$

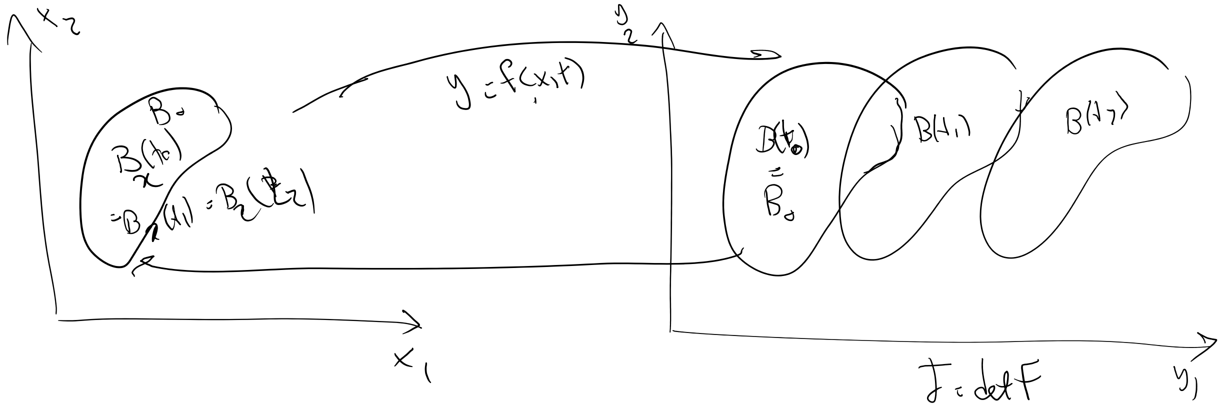
Formal proof of Reynold's transport theorem



Theorem 145 (Transport Theorem) *Let $g \in C^1(\Im, \Re)$ be a spatial scalar field. Then*

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{P}_t} g(\mathbf{y},t)dV_y &= \int_{\mathcal{P}_t}\left[\frac{\partial g}{\partial t}(\mathbf{y},t) + g_{,i}(\mathbf{y},t)\hat{v}_i(\mathbf{y},t) + g(\mathbf{y},t)\hat{v}_{i,i}(\mathbf{y},t)\right]dV_y \\ &= \int_{\mathcal{P}_t}\left\{\frac{\partial g}{\partial t}(\mathbf{y},t) + [g\hat{v}_i]_{,i}(\mathbf{y},t)\right\}dV_y \\ &= \int_{\mathcal{P}_t}\frac{\partial g}{\partial t}(\mathbf{y},t)dV_y + \int_{\partial\mathcal{P}_t} g(\mathbf{y},t)\left[\hat{\mathbf{v}}(\mathbf{y},t)\cdot\mathbf{n}(\mathbf{y},t)\right]dA_y,\end{aligned}$$

where $\mathbf{n}(\mathbf{y},t)$ is the outward unit normal to $\partial\mathcal{P}_t$ at $\mathbf{y}$.[20]



$J = \det F$

$$G(t) = \int_{B(t)} g(y,t)\, dV_y = \underbrace{\int_{B_0} g(y(x,t),t)\left(J\, dV_x\right)}_{\text{excellent! the domain is not moving in time}}$$

$$\frac{DG}{Dt}(t) = \frac{D}{Dt}\left(\int_{B} g(y(x,t),t)\, J(x,t)\, dV_x\right)$$

material

$$\left(\frac{D}{Dt}\right)\int^{1} f(x,t)\, dx$$

$\mathrm{Dt} \qquad \mathrm{Dt} \int_{B_0} g(\qquad X$ $\qquad \left(\frac{D}{Dt}\right)\int_0 \to f(x,t)\,dx$

because the domain is fixed $\overset{\text{material}}{\vee}$ time derivative goes inside

$$\frac{DG}{Dt}(t) = \int_{B_0} \frac{D}{Dt}\left(g J\right) dV_X = \int_{B_0}\left(\left(\frac{Dg}{Dt}\right) J + g\frac{DJ}{Dt}\right) dV_X \quad ①$$

$$\frac{Dg}{Dt} = \left.\frac{\partial g}{\partial t}\right|_{y \text{ fixed}} + \left(\nabla_y g\right) v \quad ②$$

recall $g(y,t) = g(y(x,t),t)$

from last time

$$\frac{DJ}{Dt} = \underbrace{\left(\nabla_y \cdot v\right)}_{\text{div } v} J \quad ③$$

plug ② & ③ into ①

$$\frac{DG}{Dt} = \int_{B_0} J\Bigg(\underbrace{\frac{\partial g}{\partial t} + \left(\nabla_y g\right) v}_{\frac{Dg}{Dt}} + \underbrace{g\left(\nabla_y \cdot v\right) J}_{\frac{DJ}{Dt}}\Bigg) dV_y$$