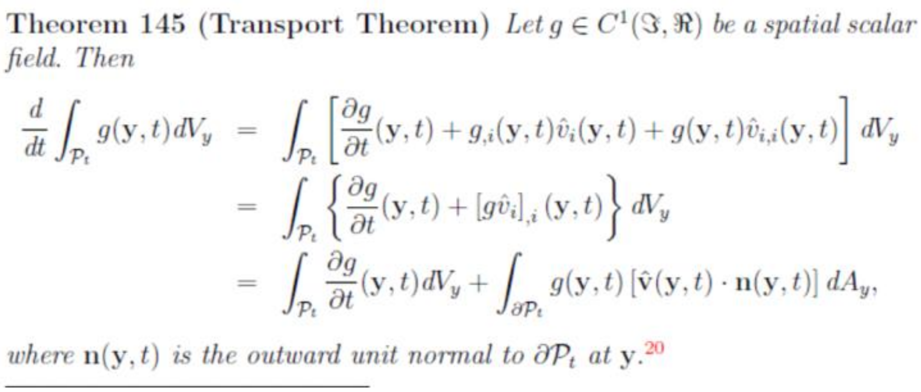
CM2023/11/15

Wednesday, November 15, 2023 9:10 AM



Theorem 145 (Transport Theorem) *Let $g \in C^1(\Im, \Re)$ be a spatial scalar field. Then*

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{P}_t} g(\mathbf{y},t)dV_y &= \int_{\mathcal{P}_t}\left[\frac{\partial g}{\partial t}(\mathbf{y},t) + g_{,i}(\mathbf{y},t)\hat{v}_i(\mathbf{y},t) + g(\mathbf{y},t)\hat{v}_{i,i}(\mathbf{y},t)\right]dV_y \\ &= \int_{\mathcal{P}_t}\left\{\frac{\partial g}{\partial t}(\mathbf{y},t) + \left[g\hat{v}_i\right]_{,i}(\mathbf{y},t)\right\}dV_y \\ &= \int_{\mathcal{P}_t}\frac{\partial g}{\partial t}(\mathbf{y},t)dV_y + \int_{\partial\mathcal{P}_t} g(\mathbf{y},t)\left[\hat{\mathbf{v}}(\mathbf{y},t)\cdot\mathbf{n}(\mathbf{y},t)\right]dA_y,\end{aligned}$$

where $\mathbf{n}(\mathbf{y},t)$ is the outward unit normal to $\partial\mathcal{P}_t$ at $\mathbf{y}$.[20]

ln -tm

$$\frac{DG}{Dt}(t) = \frac{D}{Dt}\int_{B_0} g(\underbrace{y(x,t)},t)\,J(x,t)\,dV_x \qquad \frac{D}{Dt}\int_0^1 f(x,t)\,dx$$

because the domain is fixed material time derivative goes inside

$$\frac{DG}{Dt}(t) = \int_{B_0}\frac{D}{Dt}(gJ)\,dV_x = \int_{B_0}\left(\left(\frac{Dg}{Dt}\right)J + g\frac{DJ}{Dt}\right)dV_x \quad ①$$

$$\frac{Dg}{Dt} = \left.\frac{\partial g}{\partial t}\right|_{y\text{ fixed}} + (\nabla_y g)\,v \quad ②$$

recall $g(y,t) = g(y(x,t),t)$

from last time

$$\frac{DJ}{Dt} = \underbrace{(\nabla_y \cdot v)}_{\text{div } v}J \quad ③$$

plug ② & ③ into ①

$$\frac{DG}{Dt} = \int_{B_0} J\underbrace{\left(\frac{\partial g}{\partial t} + (\nabla_y g)v\right)}_{\frac{Dg}{Dt}} + \underbrace{g(\nabla_y\cdot v)J}_{\frac{DJ}{Dt}}\,dV_x \qquad dV_y = J\,dV_x$$

$$\boxed{\frac{DG}{Dt} = \int_B\left(\frac{\partial g}{\partial t} + \underbrace{(\nabla_y g)v + g(\nabla_y\cdot v)}_{*}\right)dV_y \qquad \text{L1 below}}$$

④

$\star$

B

①

$$\frac{d}{dt}\int_{\mathcal{P}_t} g(\mathbf{y},t)dV_y = \int_{\mathcal{P}_t}\left[\frac{\partial g}{\partial t}(\mathbf{y},t) + g_{,i}(\mathbf{y},t)\hat{v}_i(\mathbf{y},t) + g(\mathbf{y},t)\hat{v}_{i,i}(\mathbf{y},t)\right]dV_y \qquad L1$$

to get to full divergence form L2:

Let's assume g is a scalar

$$\star \qquad \underbrace{(\nabla_y g)v + g(\nabla_y \cdot v)}_{\substack{\nabla_y \cdot \\ \mathrm{div}(gv)}} = \frac{\partial g}{\partial y_i} v_i + g\frac{\partial v_i}{\partial y_i} = \frac{\partial g v_i}{\partial y_i} =$$

Same can be done for any order tensor g

① $\Longrightarrow$

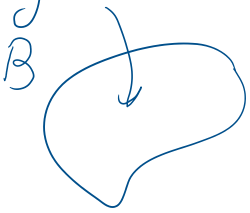
$$\frac{DG}{Dt} = \int_{B(t)} \left(\frac{\partial g}{\partial t} + \nabla_y \cdot gv\right) dV_y$$

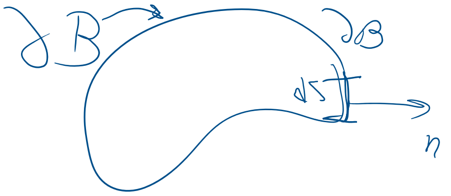
$$= \int_{\mathcal{P}_t}\left\{\frac{\partial g}{\partial t}(\mathbf{y},t) + [g\hat{v}_i]_{,i}(\mathbf{y},t)\right\}dV_y \qquad L2$$

②

to get the boundary integral, we'll use the divergence theorem

$$\int_B f_{,i}\, dV = \int_{\partial B} f n_i\, dS$$





$$\vec{dS} = \vec{n}\, dS$$

$$\text{★} \quad \int_B \underset{\substack{\text{vector or} \\ \text{higher order tensor}}}{\nabla \cdot \overset{}{f}}\, dV = \int_{\partial B} f \cdot \underbrace{n\, dS}_{\vec{dS}}$$

Divergence theorem

Use ★ in ②

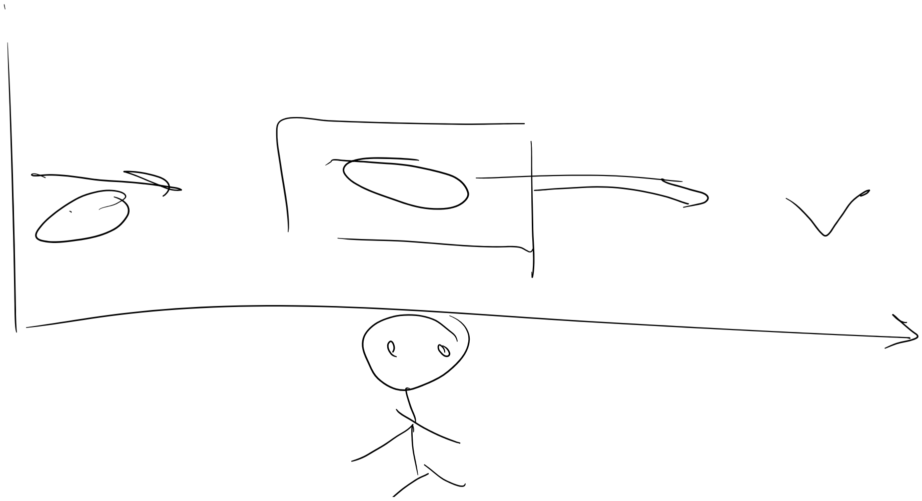
$$\frac{DG}{Dt} = \int_{B(t)} \left(\frac{\partial g}{\partial t} + \nabla_y \cdot \overset{f \text{ above}}{\widetilde{gv}} \right) dV_y \implies$$

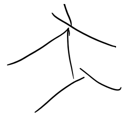
$$\frac{DG}{Dt} = \int_{B(t)} \frac{\partial g}{\partial t} + \int_{\partial B(t)} g v \cdot n\, dS_y \qquad ③$$

Connection

L3

$$= \int_{\mathcal{P}_t} \frac{\partial g}{\partial t}(\mathbf{y}, t)\, dV_y + \int_{\partial \mathcal{P}_t} g(\mathbf{y}, t) \left[\hat{\mathbf{v}}(\mathbf{y}, t) \cdot \mathbf{n}(\mathbf{y}, t) \right] dA_y,$$





What's a balance law?

$$G = \int_{B(t)} g \, dV$$

we want to see how G evolves in time?

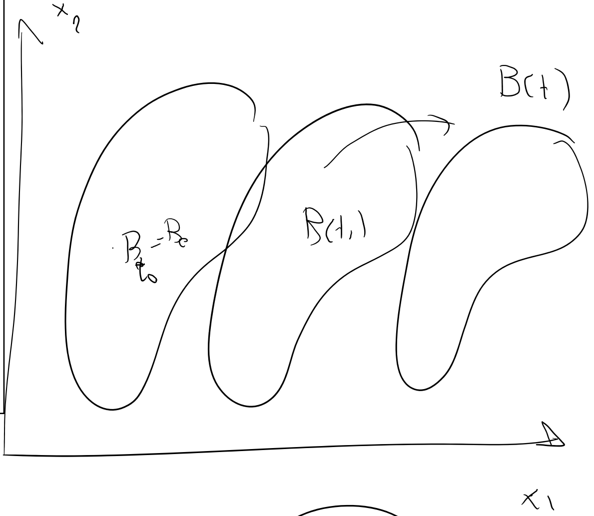
④

$$\underbrace{\frac{DG}{Dt}} = \int_{B(t)} \underbrace{r_g}_{\text{source term for } g} dV - \int_{\partial B(t)} f_g^y \cdot n_y \, dS_y$$

"diffuse"

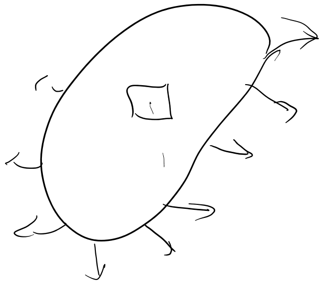
outward spatial flux density of G

General Balance law



Balance of linear momentum

$$\sum F = \frac{D}{D} \underline{\text{Linear moment}}$$



$$r_g = \rho b$$

body force

$$\int_{\partial B} \vec{t} \, dS_y = \int \sigma \cdot n \, dS_y$$

$$\text{RHS} \quad \int \underbrace{\rho b}_{\text{source term}} dV_y - \int \underbrace{(-\sigma)} \cdot n \, dS_y$$

$-\sigma$ = outward spatial linear momentum density

Energy only considering heat conducti

$$\underset{\text{energy}}{\frac{D}{Dt}\mathcal{E}} = \int_{B(t)} \underset{\text{head source}}{Q}\, dV - \int_{\partial B(t)} \underset{\text{heat flux density}}{q}\, n\, dS$$

Eqn ④
$$\frac{DG}{Dt} = \frac{D}{Dt}\int_{B(t)} g\, dV_y = \int_{B(t)} r_g\, dV_y - \int_{\partial B(t)} f^y_{dg} \cdot d\vec{S}_y \qquad d\vec{S}_y = n_y\, dS_y$$

Eqn ③
$$\frac{DG}{Dt} = \overbrace{\int_{B(t)} \frac{\partial g}{\partial t}\, dV_y + \int_{\partial B(t)} \underbrace{g \otimes v}_{\text{dyadic product}} \cdot d\vec{S}_y}$$

$$g\,(v \cdot n) = (g \otimes v) \cdot n$$

$$\underbrace{\int_{B(t)} \frac{\partial g}{\partial t}\, dV_y + \int_{\partial B(t)} g \otimes v \cdot d\vec{S}_y}_{\frac{DG}{Dt}} = \int_{B(t)} r_g\, dV_y - \int_{\partial B(t)} f^y_{dg} \cdot d\vec{S}_y$$

⑤
$$\boxed{\int_{B(t)} \frac{\partial g}{\partial t}\, dV_y + \int_{\partial B(t)} \Big(\underbrace{g \otimes v}_{\substack{\text{conductive}\\ \text{spatial flux}\\ \text{density}}} + \underbrace{f^y_{dg}}_{\substack{\text{“diffuse”}\\ \text{spatial flux density}}}\Big) \cdot d\vec{S}_y = \int_{B(t)} r_g\, dV_y}$$

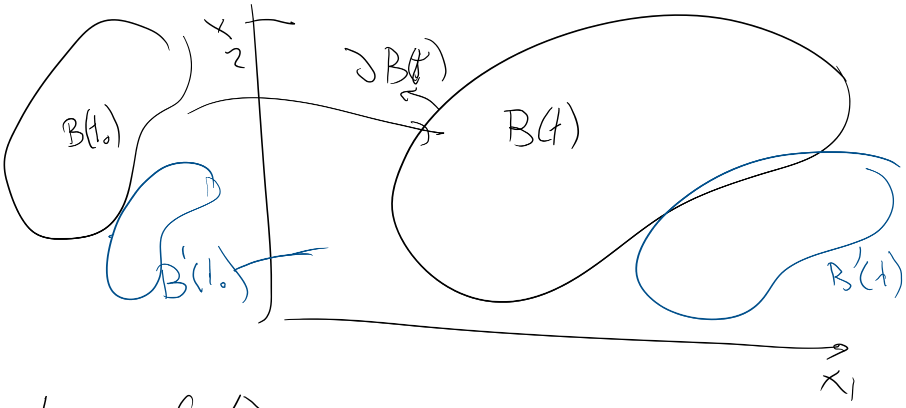
Balance laws provide:

1. Partial differential equations (PDEs)
2. Jump conditions (useful e.g. in determining shock propagation speed, hydraulic jumps, …)

PDE: Apply divergence theorem to eqn ⑤ (2nd term)

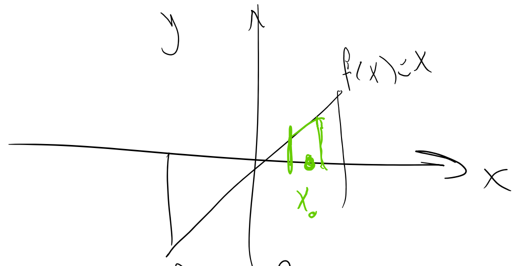
$$\int \frac{\partial g}{\partial t}\, dV_y + \int \nabla_y \cdot \left(g \otimes v + f^y_{dg}\right) dV_y = \int r_g\, dV_y$$

$$\int_{B(t)} \frac{\partial g}{\partial t}\, dV_y + \int_{B(t)} \nabla_y \cdot \left(g \otimes v + f^{y}_{\partial g} \right) dV_y = \int_{B(t)} r_g\, dV_y$$



$$\forall_{\text{any}\, B(t)} \int_{B(t)} \left(\frac{\partial g}{\partial t} + \nabla_y \cdot \left(g \otimes v + f^{y}_{\partial g} \right) - r_g \right) dV_y = 0 \qquad ⑥$$

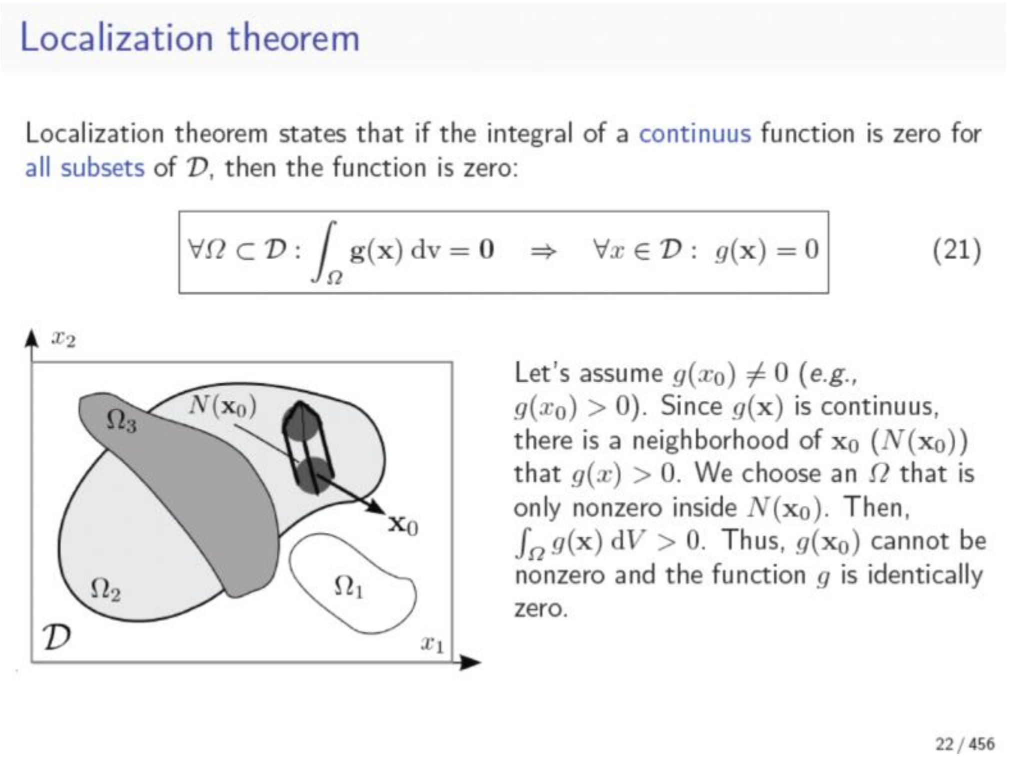
1D



$$\int_{-1}^{1} f(x)\, dx = 0$$

$$\& \; f(x) \neq 0$$

However if we say that the integral (of a continuous function) f is zero over all sets then f = 0



Localization theorem

Localization theorem states that if the integral of a continuus function is zero for all subsets of $\mathcal{D}$, then the function is zero:

$$\boxed{\forall \Omega \subset \mathcal{D}: \int_{\Omega} \mathbf{g}(\mathbf{x})\, \mathrm{dv} = \mathbf{0} \quad \Rightarrow \quad \forall x \in \mathcal{D}: \; g(\mathbf{x}) = 0} \qquad (21)$$

Let's assume $g(x_0) \neq 0$ (*e.g.*, $g(x_0) > 0$). Since $g(\mathbf{x})$ is continuus, there is a neighborhood of $\mathbf{x}_0$ ($N(\mathbf{x}_0)$) that $g(x) > 0$. We choose an Ω that is only nonzero inside $N(\mathbf{x}_0)$. Then, $\int_{\Omega} g(\mathbf{x})\, dV > 0$. Thus, $g(\mathbf{x}_0)$ cannot be nonzero and the function g is identically zero.

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PDE

Applying localizatin to eqn ⑥ we'll get the PDE

(temporal flux density)

$$\underbrace{\frac{\partial g}{\partial t} + \nabla_y \cdot f_g^y - r_g}_{} = 0$$

→ source term for G

⑥

$$f_g^y = \underbrace{g \otimes v}_{} + f_{dg}^y$$

"total" spatial flux density for G

its convective part

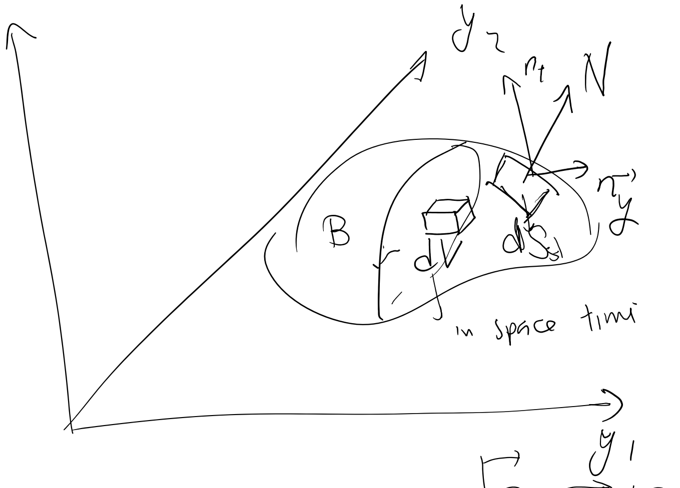
its "diffuse" part

Can we express ⑥ & in general Balance laws in space time

$$\underbrace{F_g}_{\text{spacetime flux density}} = \begin{bmatrix} f_g^y \\ g \end{bmatrix}$$

→ spatial flux density

→ temporal flux density



$$d\vec{S}_{st} = \vec{N}\, dS_{st}$$

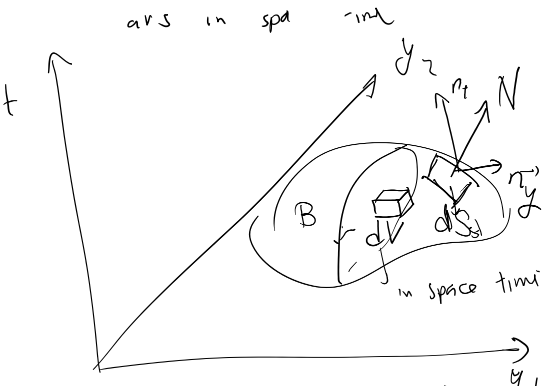
$$\underbrace{\vec{N}}_{\text{normal in spacetime}} = \begin{bmatrix} \vec{n}_y \\ n_t \end{bmatrix}$$

Balance law in spacetime says $\forall B$ in spacetime

$$\boxed{\int_{\partial B} F_g \cdot d\vec{S}_{st} = \int_B r_g \, dV_{st}}$$

Ⓘ

that is $$\underbrace{\int_{\partial B} F_g \cdot N \, dS_{st}}_{\text{Apply divergence theorem for this term}} = \int_B r_g \, dV_{st}$$



$$\vec{dS}_{st} = \vec{N} dS_{st}$$

$$\int_B \nabla_{st} \cdot F_g \, dV_{st} = \int_B r_g \, dV_{st}$$

$$\Rightarrow \int_B \underbrace{(\nabla_{st} \cdot F_g - r_g)} dV_{st} = 0 \quad \forall B \text{ in space time}$$

$$\underbrace{\nabla_{st} \cdot F_g} - r_g = 0 \qquad \nabla_{st} \cdot \begin{bmatrix} f_g^y \\ g \end{bmatrix} - r_g = 0 \Rightarrow$$

$f_g^y \rightarrow$ spatial

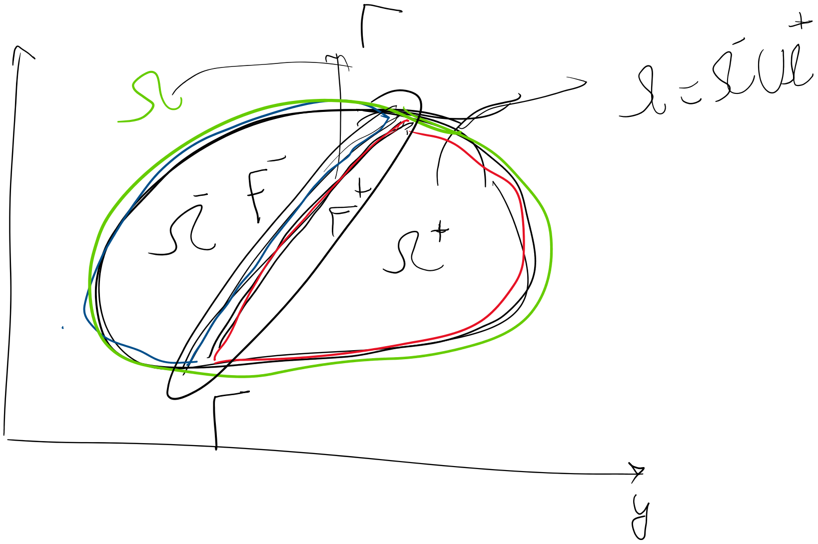
$g \rightarrow$ temporal

$$\underbrace{\nabla_y \cdot f_g^y} + \frac{\partial g}{\partial t} - r_g = 0$$ this is eqn ⑥

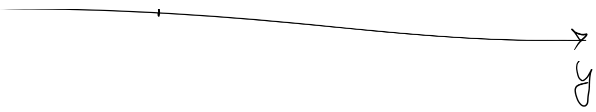
Hint about your HW example

i) $$\int_{\partial\Omega^-} F \, dS - \int_{\Omega^-} r \, dV = 0$$

ii) $$\int_{\partial\Omega^+} F \, dS - \int_{\Omega^+} r \, dV = 0$$



$\partial\Omega^+$ Ω^+



iii) $\int_{\Omega} F\,ds - \int_{\Omega} r\,dV = 0$

iii) − i) − ii) . . .

$$\int_{\Gamma} (F^+ - F^-)\cdot \underbrace{d\vec{S}}_{N\,dS} = 0$$

let $\Gamma \to 0$

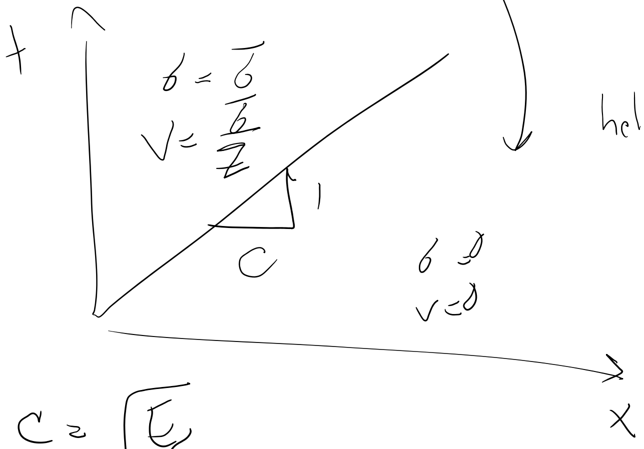
Jump condition $\boxed{(F^+ - F^-)\cdot N = 0}$ ⑧

holds

This is also called the Rankine-Hugoniot jump condition

Prob

$Z = \sqrt{E\rho}$



$c = \sqrt{E/\rho}$

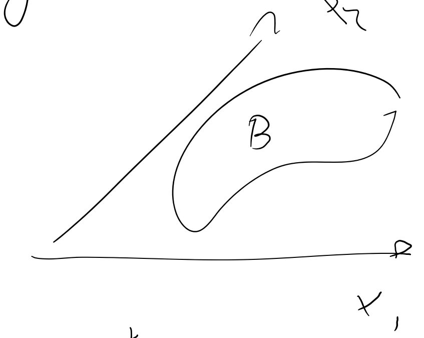
Static / steady state balance laws:
No change versus time

$$\cancel{\frac{DG}{Dt}} = \int_{B(t)} \underset{\text{source term}}{\underbrace{r_g}}\, dV - \int_{\partial B(t)} \underset{\text{"diffuse"}}{f_g^y} \cdot n_y\, dS_y$$

$\wedge^{x_2}$

eq4

Static $$\int_{\partial B} f^{y}_{dg} \cdot n_y \, dS_y = \int_{B} r_g \, dV$$



eqn ?
dynamic
in
spacetime $$\int_{\partial B} F_g \cdot d\vec{S}_{st} = \int_{B} r_g \, dV_{st}$$