CM2023/11/20

Monday, November 20, 2023 9:40 AM

Balance of mass:

general statement of balance law from last time

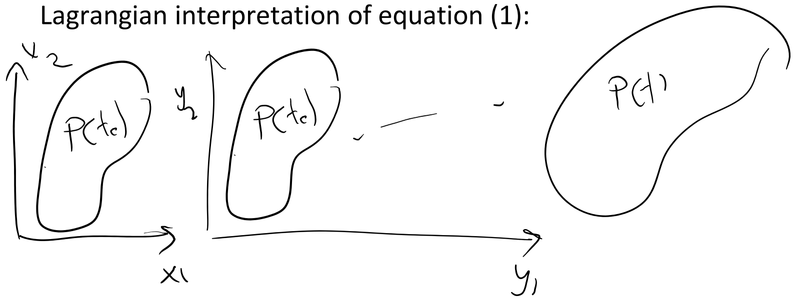
$$\frac{DG}{Dt} = \frac{D}{Dt}\int_{P(t)} g\, dV_y = \underbrace{\int_{P(t)} \hat{g}^y\, dV_y}_{\text{source term}} - \underbrace{\int_{\partial P(t)} f^y_{dg}\, dS_y}_{\text{diffuse spatial flux}}$$

Mass $\quad G = M \quad g = \frac{\text{Mass}}{\text{Volume}} = \rho$

for balance of mass $\quad$ source term $= 0 \quad\quad 0$

$$\boxed{\frac{D}{Dt}\overbrace{\int_{P(t)} \rho\, dV_y}^{M} = 0 \quad \text{Conservation of mass}} \quad (1)$$

Lagrangian interpretation of equation (1):

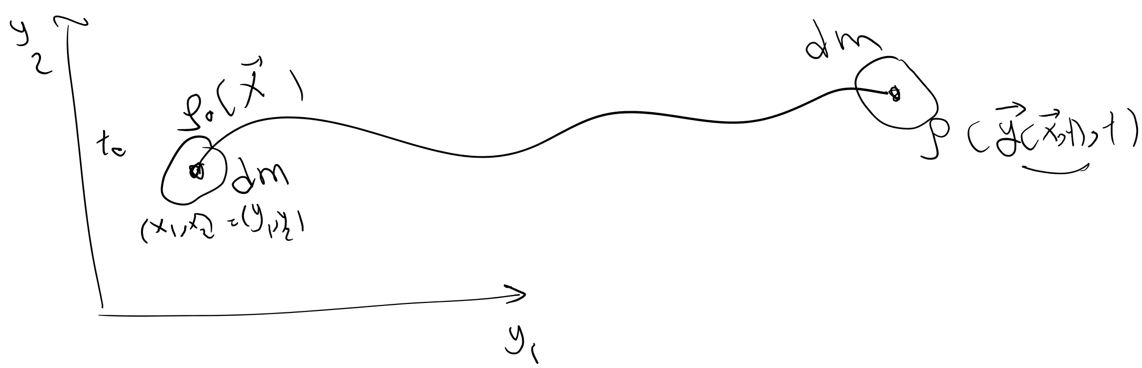


$$\frac{D}{Dt}M = 0 \Longrightarrow M(t_0) = M(t_1) \Rightarrow \overbrace{\int_{P(t_0)} \underbrace{\rho_0}_{\text{density at initial configuration}}(x)\, dV_x}^{M(t_0)} = \overbrace{\int_{P(t)} \rho(y,t)\, dV_y}^{M(t)}$$

$$\therefore \overbrace{\int_{P(t_0)} \rho(y(x,t),t)\underbrace{(J\, dV_x)}_{dV_y}}^{\text{pull-back of } \int \rho(y,t)\, dV_y} = \int_{P(t_0)} \rho_0(x)\, dV_x$$

$$\Longrightarrow \int_{P(t_0)} (\rho J - \rho_0)\, dV_X = 0$$ since this holds for any $P(t_0)$ we'll use localization to show:

$$\boxed{\rho J = \rho_0 \qquad \underbrace{\rho(y(x,t),t)}_{\rho @ \text{ time } t} = \frac{\overbrace{\rho_0(X)}^{\rho @ t_0}}{J(X,t)}}$$ ②



Basically eqn ② says dm is "fixed"

@ time t_0 $\quad dm = \rho_0(x)\, dV_X$

@ time t $\quad dm = \rho(y,t)\, dV_y = \rho\big(y(x,t),t\big) J\, dV_X$

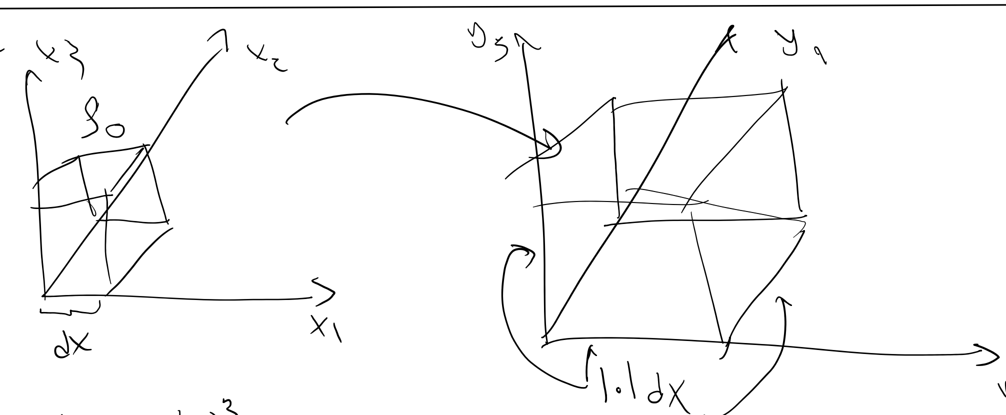
$\Rightarrow dm = \rho_0(X)\, \cancel{dV_X} = \rho(y(X,t),t)\; J(X,t)\, \cancel{dV_X}$

we're back to eqn ②

Numerical example

$y_i = 1.1 x_i$

@ final time t



$dV_X = (dx)^3$

$dV_y = (1.1dx)^3 = 1.1^3 (dx)^3$

$dV_x = (dx)^3$

$dV_y = (1.1\,dx)^3 = 1.1^3 (dx)^3$

$= (1.1)^3 dV_x$

$dm = dV_x \rho_0$

$= dV_y \rho = (1.1)^3\, dV_x$

$\Rightarrow \quad \rho = \frac{\rho_0}{\underbrace{(1.1)^3}_{J}} = \frac{\rho_0}{J}$

We can use the fact that dm does not change in fast derivation of many equations:
Example: the "Reduced Reynolds transport theorem

Theorem 151 (Reduced Transport Theorem) *Let $g \in C^1(\Im, \Re)$. Then*

$$\frac{d}{dt}\int_{\mathcal{P}_t} g(\mathbf{y},t)\rho(\mathbf{y},t)dV_y = \int_{\mathcal{P}_t}\left[\frac{\partial g}{\partial t}(\mathbf{y},t) + g_{,i}(\mathbf{y},t)\hat{v}_i(\mathbf{y},t)\right]\rho(\mathbf{y},t)dV_y.$$

$$= \int_{\mathcal{P}_t}\left[\frac{\partial g}{\partial t}(\mathbf{y},t) + \nabla g(\mathbf{y},t)\cdot\hat{\mathbf{v}}(\mathbf{y},t)\right]\rho(\mathbf{y},t)dV_y$$

extra rel to transport theorem

Theorem 145 (Transport Theorem) *Let $g \in C^1(\Im, \Re)$ be a spatial scalar field. Then*

$$\frac{d}{dt}\int_{\mathcal{P}_t} \tilde{g}(\mathbf{y},t)dV_y = \int_{\mathcal{P}_t}\left[\frac{\partial \tilde{g}}{\partial t}(\mathbf{y},t) + \tilde{g}_{,i}(\mathbf{y},t)\hat{v}_i(\mathbf{y},t) + \tilde{g}(\mathbf{y},t)\hat{v}_{i,i}(\mathbf{y},t)\right]dV_y$$

missing in Red. Tr. theorem ☆

$$\frac{Dg}{Dt}dV_y + g\frac{D\,dV_y}{Dt} = \int_{\mathcal{P}_t}\left\{\frac{\partial \tilde{g}}{\partial t}(\mathbf{y},t) + [\tilde{g}\hat{v}_i]_{,i}(\mathbf{y},t)\right\}dV_y$$

$$= \int_{\mathcal{P}_t}\frac{\partial \tilde{g}}{\partial t}(\mathbf{y},t)dV_y + \int_{\partial\mathcal{P}_t}\tilde{g}(\mathbf{y},t)\left[\hat{\mathbf{v}}(\mathbf{y},t)\cdot\mathbf{n}(\mathbf{y},t)\right]dA_y,$$

There is a formal proof of Red. transport theorem using theorem 145 (transport theorem) by using

$\tilde{g} = \rho g$ plug ρg in ☆ and show ☆ will result in 151 after some manipulation

suggested for HW8

$$\frac{D}{Dt}\int_{P(t)} g\rho\,\overbrace{dV_y}^{dm} = \frac{D}{Dt}\int_{P(t)} g\,dm = \int_{P(t)}\left(\frac{Dg}{D}\,dm + g\,\underbrace{\frac{D\,dm}{Dt}}_{0}\right)$$

$\int \frac{Dg}{\ldots}\,(\rho\,\overbrace{dV_y}^{dm}) \quad \int(\frac{Dg}{\ldots}\ldots)\ldots$

$$= \int_{P(t)} \underbrace{\frac{Dg}{Dt}}\,(\overbrace{\rho\, dV_y}^{dm}) = \int \underbrace{\left(\frac{\partial g}{\partial t}\Big|_y + (\nabla_y g)\, v\right)}_{\frac{Dg}{Dt}} \rho\, dV_y$$

$$\frac{d}{dt}\int_{\mathcal{P}_t} g(\mathbf{y},t)\rho(\mathbf{y},t)\,dV_y = \int_{\mathcal{P}_t}\left[\frac{\partial g}{\partial t}(\mathbf{y},t) + g_{,i}(\mathbf{y},t)\hat{v}_i(\mathbf{y},t)\right]\rho(\mathbf{y},t)\,dV_y.$$

Conservation of mass in the Eulerian framework

eq (4):

$$\frac{DM}{Dt} = \underbrace{\frac{D}{Dt}\int_{P(t)} \rho\, dV_y}_{\text{need to calculate this}} = 0$$

$$\int_{P(t)} \frac{\partial \rho}{\partial t}\Big|_y dV_y + \int_{\partial P(t)} \rho\, v \cdot \underbrace{n_y\, dS_y}_{d\vec{S}_y} = 0$$

$$\int_{P(t)} \frac{\partial \rho}{\partial t}\Big|_y dV_y + \int_{P(t)} \nabla_y \cdot (\rho v)\, dV_y = 0$$

$$\Rightarrow \int_{P(t)} \left(\frac{\partial \rho}{\partial t} + \nabla_y \cdot (\rho v)\right) dV_y = 0$$

since $P(t)$ can be arbitrary we'll use the localization theorem to get

(3)
$$\boxed{\begin{gathered}\left.\frac{\partial \rho(y,t)}{\partial t}\right|_y + \nabla_y \cdot \big(\rho(y,t)\, v(y,t)\big) = 0 \\ \left.\frac{\partial \rho}{\partial t}\right|_y + \operatorname{div}(\rho v) = 0 \\ \text{Balance of mass in Eulerian framework}\end{gathered}}$$

Having a full divergence in the strong form (PDE) is good because:

1. The term that div acts on is the spatial flux density (div acts on spatial flux density)
2. Very suitable for the formulation of many numerical methods (FEM, discontinuous Galerkin)

However, there is another way to write the balance of mass

$$\left.\frac{\partial \rho}{\partial t}\right|_y + \nabla_y\cdot(\rho v) = \left.\frac{\partial \rho}{\partial t}\right|_y + \frac{\partial \rho v_i}{\partial y_i} = \underbrace{\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial y_i} v_i}_{\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \nabla_y \rho\, v} + \rho \underbrace{\frac{\partial v_i}{\partial y_i}}_{\nabla_y \cdot v}$$

Recall:

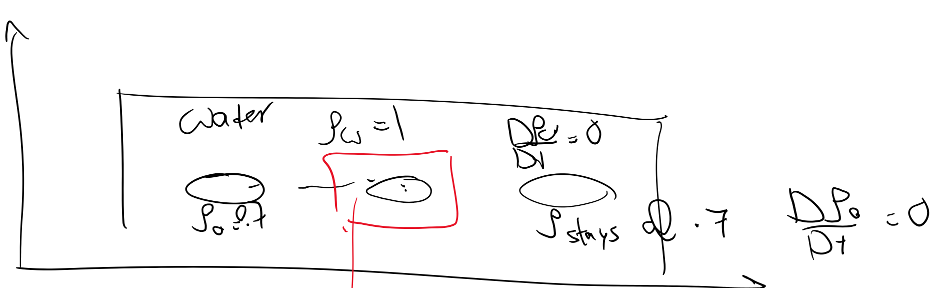
$$\frac{Dg}{Dt} = \frac{\partial g}{\partial t} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{Dg}{Dt} = \frac{\partial g}{\partial t} + (\nabla_y g)\, v$$

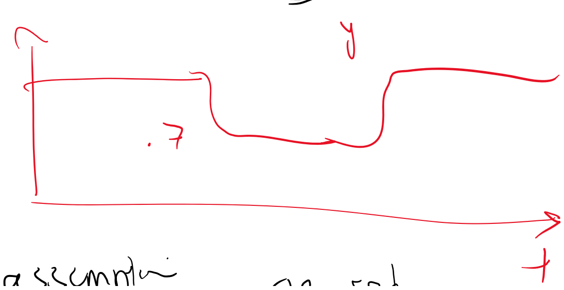
Balance of mass

$$\boxed{\left.\frac{\partial \rho}{\partial t}\right|_y + \nabla_y\cdot(\rho v) = \frac{D\rho}{Dt} + \rho\, \nabla_y \cdot v = 0} \quad (4)$$

Incompressible fluid assumption



$$0 + \left.\frac{\partial \rho}{\partial t}\right|_y$$



C[illegible]ible fluid assumption [illegible] for incompressible fluid assumption we set

$\frac{D\rho}{Dt} = 0 \implies$

(5)
$$\boxed{\begin{aligned} &\operatorname{div} V = 0 \\ &\nabla_y \cdot V = 0 \end{aligned} \qquad \text{incompressibility condition}}$$

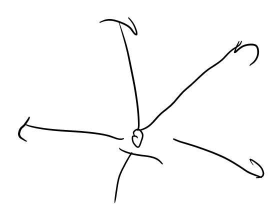
Summary of all conservation of mass equations:

	General eqn	incompressible
Lagrangian	$\rho(x,t) = \frac{\rho_0(X)}{J(x,t)}$	$J(x,t)=1 \quad \rho(x,t) = \rho_0(x)$
Eulerian	$\frac{\partial \rho(y,t)}{\partial t} + \operatorname{div}(\rho(y,t) V(y,t)) =$ $\frac{D\rho(y,t)}{Dt} + \rho(y,t) \operatorname{div}(V(y,t)) = 0$	$\operatorname{div}(V(y,t)) =$ $\frac{\partial v_1}{\partial y_1} + \frac{\partial v_2}{\partial y_2} + \frac{\partial v_3}{\partial y_3} = 0$

(6)

Balance of linear momentum:

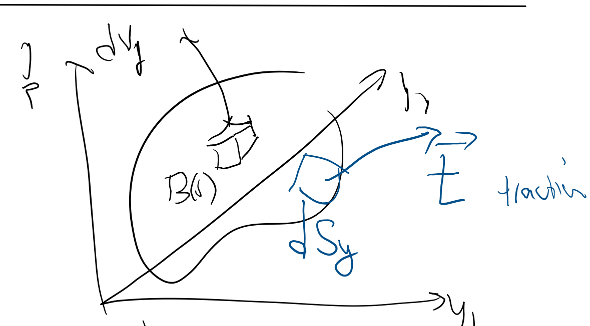
$$\frac{D}{Dt}\Big(\underbrace{\text{linear momentum}}_{P}\Big) = \overbrace{\sum F}^{\text{sum of forces}}$$



for a particle $P = m\vec{V}$

for a continuum

(7) $$P = \int_{B(t)} \vec{V} \underbrace{dm}_{\rho\, dV_y} = \int_{B(t)} \underbrace{\rho v}_{p}\, dV_y$$ or m (fluid mechanics)



$\downarrow$ $\mathcal{B}(t)$ ... $\mathcal{B}(t)$ ... or m (fluid mechanics)

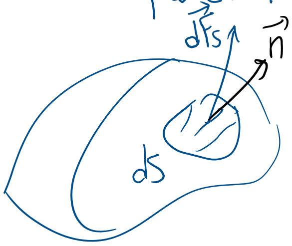
linear momentum

p (or m) = (temporal) linear momentum density

$$⑧ \quad \frac{DP}{Dt} = \sum F = \int_{B(t)} \underbrace{\vec{b}}_{\substack{\text{force} \\ \text{per mass}}} \, d\underbrace{m}_{\rho dV_y} + \int_{\partial B(t)} \vec{t} \, dS_y$$

e.g. for gravitational force $\vec{b} = -\underbrace{g}_{\substack{\text{gravitational} \\ \text{acceleration}}} \vec{e}_z$

traction = density of force per surface



$$\vec{t} = \lim_{dS \to 0} \frac{d\vec{F_s}}{dS}$$

$\vec{n}$: normal vector

anytime you have something like $\vec{F}$ integrated over the surface we can find a 1-order higher tensor $\ni$

$$⑨ \quad \underbrace{\vec{t}}_{\text{vector}} = \underbrace{\sigma}_{\substack{\text{2nd order} \\ \text{stress} \\ \text{tensor}}} \underbrace{n}_{\substack{\text{normal} \\ \text{vector}}}$$

from ⑦ ⑧, ⑨ :

$$⑩ \quad \boxed{\frac{DP}{Dt} = \frac{D}{Dt} \int_{B(t)} \underbrace{\overset{p}{\rho v}}_{g} \, dV_y = \int_{B(t)} \underbrace{\rho b}_{\substack{r_g^y \\ \text{source term}}} \, dV_y + \int_{\partial B(t)} \sigma \cdot n \, dS_y \quad f_{dp}^{y} = -\sigma}$$

compare 10 with General Balance law

$$\frac{DG}{Dt} = \frac{D}{Dt}\int_{B(t)} g \, dV_y = \int_{B(t)} r_g^y \, dV_y - \int_{\partial B(t)} f_{dg}^y \cdot n \, dS_y$$

Strong form (PDE) was

$$\left.\frac{\partial g}{\partial t}\right|_y + \nabla_y \cdot \left(g \otimes v + f_{dg}^y \right) = r_g^y$$

Bal. linear momentum: $g = p = \rho v$ $f_{dg}^y = -\sigma$ $r_g^y \to \rho b$

(11) $$\boxed{\left.\frac{\partial \rho v}{\partial t}\right|_y + \nabla_y \cdot \left(\rho v \otimes v - \sigma \right) = \rho b}$$

Equation of motion / Balance of lin momentum PDE