From the last time

Balance of linear momentum

$$(\star)\quad \frac{D}{Dt}\int_{B(t)} p\, dV_y = \int_{B(t)} \frac{\partial p}{\partial t}\, dV_y + \int_{\partial B(t)} p \otimes \vec{v}\, n_y\, dS_y = \int_{B(t)} \rho b\, dV_y + \int_{\partial B(t)} \sigma n_y\, dS_y$$

$$\int_{B(t)} \frac{\partial p}{\partial t}\, dV_y + \int_{\partial B(t)} \underbrace{\left(-\sigma + p \otimes v\right)}_{\text{total spatial flux den. for linear momentum}} n_y\, dS_y = \int_{B(t)} \rho b\, dV_y$$

Apply Div theorem

$$\forall B(t) \quad \int_{B(t)} \left(\frac{\partial p}{\partial t} + \nabla_y \cdot \left(-\sigma + p \otimes v\right) - \rho b\right) dV_y = 0$$

Apply localization

$$\frac{\partial p}{\partial t} + \nabla_y \cdot \Big(\overbrace{-\sigma}^{\text{diffusive flux}} + \overbrace{p \otimes v}^{\text{convective flux}}\Big) - \rho b = 0$$

Equation of Motion (EOM)

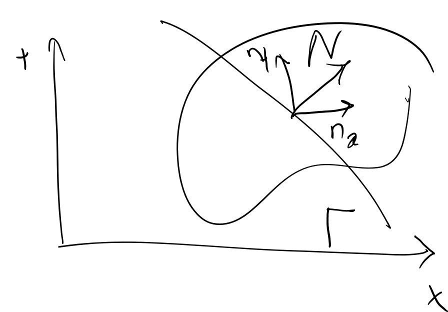
strong form of balance of linear momentum

$p = \rho \vec{v}$ (also $m = \rho \vec{v}$)

1

Next HW

$$M = \begin{bmatrix} -\sigma + p \otimes v \\ p \end{bmatrix} \begin{array}{l} \text{spatial} \\ \\ \text{temporal flux} \end{array}$$



*solid mechanics infinitesimal theory $P \cong \sigma$

$$M = \begin{bmatrix} -\sigma \end{bmatrix}$$ (P)

$$[\![M]\!] \cdot \vec{N} = 0$$

$\tau \cong 0$

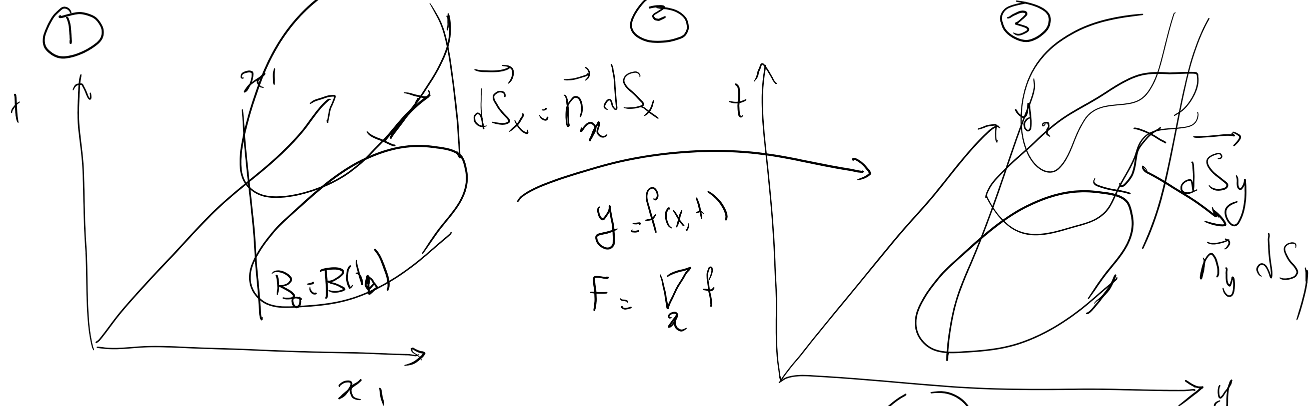
$$M = \begin{bmatrix} -\sigma \\ p \end{bmatrix} \qquad [M]\cdot \vec{N} = 0$$

check if this is correct for HW7 problem 2

$$-[\sigma]\cdot n_x + [p]\cdot n_t = 0$$

Lagrangian expression of Balance of linear momentum

(*)
$$\underbrace{\frac{D}{Dt}\int_{B(t)} p \, dV_y}_{1} = \underbrace{\int_{\partial B(t)} -\sigma\, n_y \, dS_y}_{2} + \underbrace{\int_{B(t)} \rho b \, dV_y}_{3}$$



(eqY) above is for y

term (3)

$$\int_{B(t)} \rho b \, dV_y = \int_{B_0} \rho b \,(J\, dV_x) = \int_{B_0} \underbrace{(\rho J)}_{\rho_0(x)} b \, dV_x$$

$J = \det F$

Balance of mass

$$= \int \rho_0(x)\, b \, dV_x \qquad (i)$$

domain of integration does not change in time $\Longrightarrow \frac{D}{Dt}$ goes inside

term (1)

$$\frac{D}{Dt}\int_{B(t)} p \, dV_y = \frac{D}{Dt}\int_{B_0} p J \, dV_x \Longrightarrow \int_{B_0} \frac{D}{Dt}(p J)\, dV_x$$

$\rho_0 = \rho J$

$$= \int \frac{D}{Dt}(\rho \vec{v} J)\, dV_x = \int \frac{D}{Dt}\big(\underbrace{\rho_0 \vec{v}}_{\vec{P}_o = \rho \vec{v}}\big)\, dV_x = \int \frac{D}{Dt} P_0 \, dV_x$$

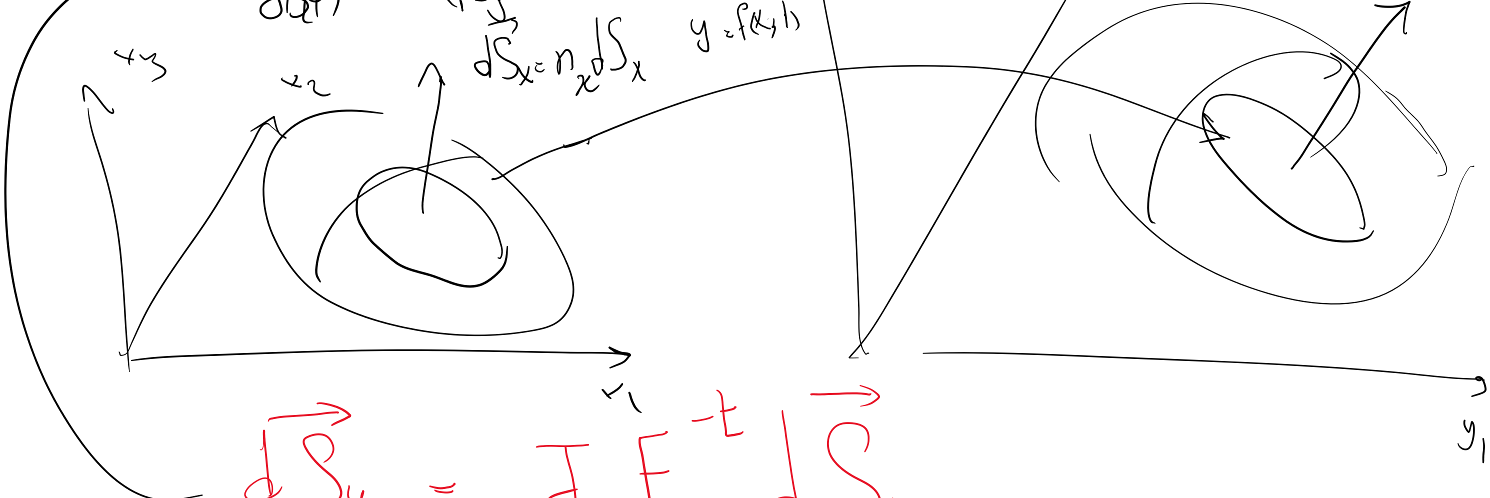
$\vec{p} = \rho \vec{v}$

$\vec{P}_o = \rho_o \vec{V}$ Lagrangian (x,t)

$\vec{p} = \rho \vec{V}$ Eulerian (y,t)

term ②

$$\int_{\partial B(t)} \sigma \cdot \underbrace{n_y \, dS_y}_{d\vec{S}_y}$$



$$d\vec{S}_y = J F^{-t} d\vec{S}_x$$

term ② :
$$\int_{\partial B_o} \sigma \left(J F^{-t} d\vec{S}_x \right) = \int_{\partial B_o} \underbrace{\left(J \sigma F^{-t} \right)}_{P} d\vec{S}_x$$

$$\boxed{P(x,t) = J(x,t)\ \sigma(y(x,t),t)\ F^{-t}(x,t)}$$

Piola-Kirchhoff stress tensor ②

Balance of linear momentum in Lagrangian coordinate

$$\frac{D}{Dt}\int_{B(t)} \rho\, dV_y = \int_{B(t)} \rho\, b\, dV_y + \int_{\partial B(t)} \sigma \cdot d\vec{S}_y$$

ρv

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\frac{D}{Dt}\int_{B_0} p_0\, dV_x = \int_{B_0} \rho_0\, b\, dV_x + \int P\, d\vec{S}_x$$

$$\parallel$$

$$\int_{B_0} \frac{D}{Dt} p_0\, dV_x$$

$\vec{p}_0 = \rho_0 \vec{v}$ referential linear momentum density

$$P = J \sigma F^{-t} \qquad (3)$$

$P \downarrow$ PK-I stress

$\sigma \downarrow$ Cauchy stress

Derive the PDE from (3):

$$\int_{B_0} \frac{D}{Dt} p_0\, dV_x = \int_{B_0} \rho_0\, b\, dV_x + \int_{\partial B_0} P\, d\vec{S}_x$$

$$\text{"} \qquad = \qquad \text{"} \qquad \int \underbrace{\nabla_x \cdot}_{\text{Div}} P\, d\vec{V}_x$$

$$\forall \vec{B}_0 : \int_{B_0} \left(\frac{D}{Dt} p_0 - \rho_0 b - \nabla_x \cdot P \right) dV_x = 0$$

$\rightarrow$ localization $\Rightarrow$ PDE shown below

B_0

$\Rightarrow$ localization $\Rightarrow$ PDE shown below

Eulerian

$$\frac{\partial}{\partial t} p + \nabla_y \cdot (-\sigma + p \otimes v) - \rho b = 0$$

Lagrangian

$$\frac{D}{Dt} P_0 \quad \nabla_x \cdot P \quad - \rho_0 b = 0$$

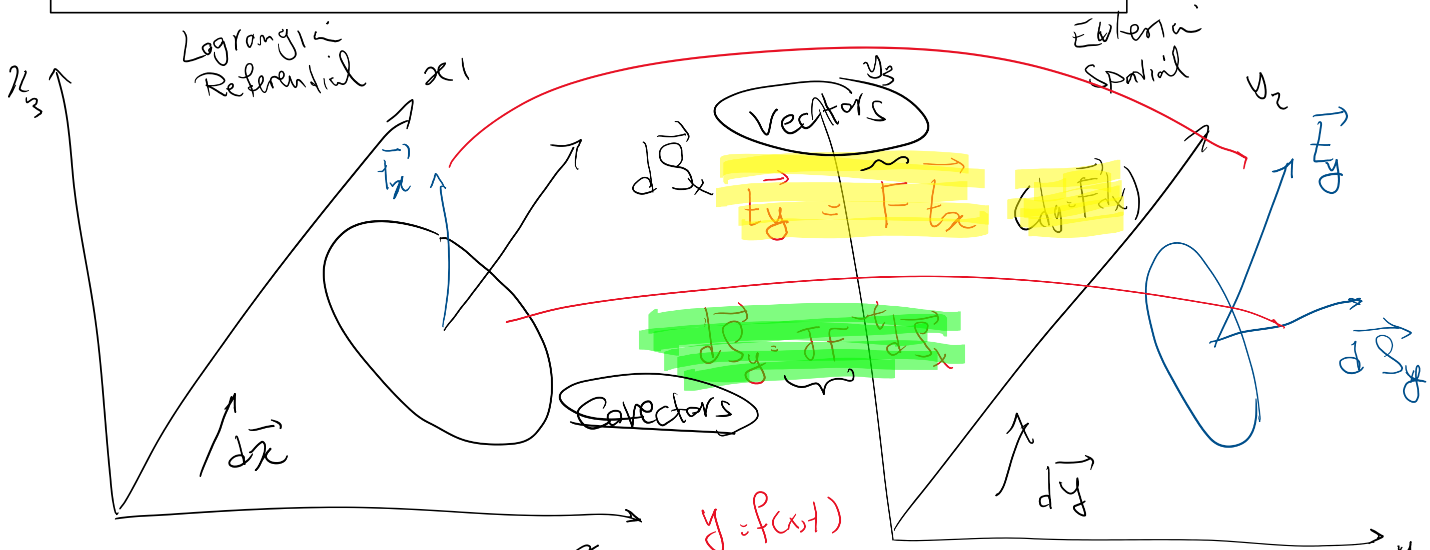
expanded form

$$P_0 = \rho_0(X)\, \vec{V}(X,t) \qquad \frac{D}{Dt} P_0 = \rho_0 \frac{DV}{Dt} = \rho_0 \frac{D^2 u}{Dt^2}$$

3D

$$\begin{cases} \rho_0 \ddot{u}_1 - (P_{11,1} + P_{12,2} + P_{13,3}) = \rho_0 b_1 \\ \rho_0 \ddot{u}_2 - (P_{21,1} + P_{22,2} + P_{23,3}) = \rho_0 b_2 \\ \rho_0 \ddot{u}_3 - (P_{31,1} + P_{32,2} + P_{33,3}) = \rho_0 b_3 \end{cases}$$

④



$y = f(x,t)$

$F = \nabla_x f$

$$\vec{t}_y = F \vec{t}_x$$

$$\vec{t}_y = \sigma \, d\vec{S}_y$$

$$\vec{t}_x = S \, d\vec{S}_x$$

$$\vec{t}_x = P \, d\vec{S}$$

$$\vec{t}_x = S \, d\vec{S}_x \qquad \underbrace{\vec{t}_y = P \, d\vec{S}_x}_{d\vec{S}_y = J F^{-t} d\vec{S}_x}$$

$$\left.\begin{array}{l} \vec{t}_y = P \, d\vec{S}_x \\ \vec{t}_y = F \, \vec{t}_x \end{array}\right\} \rightarrow F \vec{t}_x = P \, d\vec{S}_x$$

$$\rightarrow \left.\begin{array}{l} \vec{t}_x = F^{-1} P \, d\vec{S}_x \\ P = J \sigma F^{-t} \end{array}\right\} \rightarrow$$

Recall $\quad d\vec{y} = \overset{\nabla_y / x}{F} \, d\vec{x}$

$$\vec{t}_x = S \, d\vec{S}_x$$

$$\boxed{\begin{array}{c} S = J F^{-1} \sigma F^{-t} \\ \text{PK-II stress tensor} \end{array}}$$

$$S^t = J (F^{-t})^t \sigma^t (F^{-1})^t = J F^{-1} \sigma F^{-t} = S$$

$\sigma = \sigma^t$ balance of angular momentum (TAM551 notes)

S is symmetric and can be advantageous from numerical point of view

$$\boxed{\begin{array}{l} \text{EOM} : S \\ \rho_0 \dfrac{D\hat{V}}{Dt} - \nabla_x^0 \cdot (\underbrace{F S}_{P}) - \rho_0 b = 0 \end{array}} \quad (5)$$

as a side note

as a side note

$$\vec{q}_x = \left(J F^{-t} \right) \vec{q}_y$$

$$d\vec{A}_y = J F^{-t} d\vec{A}_x$$

Balance of energy:

$$E = K + U + \mathcal{E}_{EM}$$

or $\rho\varepsilon$

temporal flux of energy

heat flux density

Poynting vector

$$\underbrace{f_{dE}}_{\text{diffuse spatial flux density}} = -\sigma \cdot v + q + E \times H$$

heat source

current density

$$r_E = v \cdot \rho b + Q - \underbrace{E \cdot J}_{\text{Joulle's heating}}$$

source term

$K = \frac{1}{2} \rho v^2$ Kinetic energy density

$U = \rho \varepsilon$ internal energy density (Energy per volume)

ε = specific " " (Energy per mass)

$$\mathcal{E}_{EM} = \frac{1}{2} \left(E \cdot D + B \cdot H \right)$$