CM2023/11/29

Wednesday, November 29, 2023 9:40 AM

E temporal energy flux density

f_{dE} = diffuse spatial " " "

r_E = source term

$$\frac{D}{Dt}\int_{B(t)} E\, dV_y = \int_{B(t)} r_E\, dV_y - \int_{\partial B(t)} f_{dE}\, dS_y$$

Eulerian $\left.\frac{\partial E}{\partial t}\right|_y + \nabla_y \cdot \left(f_{dE} + E v \right) - r_E = 0$

(1) we can pull this back to Lagrangian as other balance laws

for solid mechanics with no thermal effects balance of energy is automatically satisfied

For solid mechanics, often the Lagrangian coordinate system is used. The exception can be problems with large deformations and deformation gradients for which an updated Lagrangian is favored. For solid mechanics without other effects (*e.g.*, fluid flow through the pores, *etc.*), the balance of mass automatically satisfied from (31). In the absence of other physics couplings, the balance of energy is also directly derived from the balance of linear momentum. Thus, the only relevant equation will be the balance of linear momentum (36), which expressed for $\mathbf{Y} = \mathbf{X}$ is,

$$\left.\frac{\partial \rho_0 \mathbf{v}}{\partial t}\right|_X - \nabla_X \cdot \mathbf{P} = \rho_0 \mathbf{b} \tag{38}$$

like x is course notes

1.4.2 Fluid mechanics: Navier-Stokes equations

For fluid mechanics, the balance of mass , balance of linear momentum , and balance of energy are combined to provide the *Navier-Stokes* (NS) equations in the Eulerian coordinate system,

$$\left.\frac{\partial \rho}{\partial t}\right|_x + \nabla_x \cdot \rho \mathbf{v} = 0 \quad \text{Balance of mass (continuity equation)} \tag{39a}$$

$$\left.\frac{\partial \rho \mathbf{v}}{\partial t}\right|_x + \nabla_x \cdot (\rho \mathbf{v} \otimes \mathbf{v} - \boldsymbol{\sigma}) = \rho \mathbf{b} \quad \text{Balance of linear momentum (equation of motion)} \tag{39b}$$

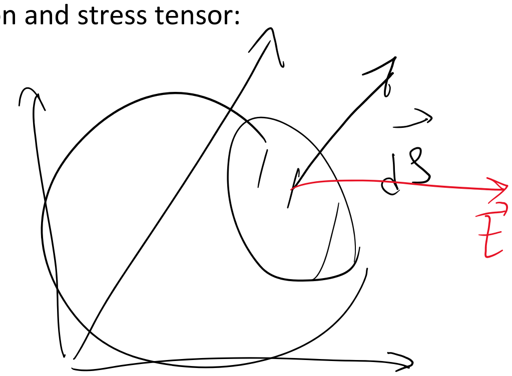
$$\left.\frac{\partial E}{\partial t}\right|_x + \nabla_x \cdot (E\mathbf{v} + \mathbf{q} - \mathbf{v}\cdot\boldsymbol{\sigma}) = \mathbf{v}\cdot\rho\mathbf{b} + Q \quad \text{Balance of energy} \tag{39c}$$

like y above

like y after

Balance of angular momentum $\longrightarrow$ σ (& S) are symmetric

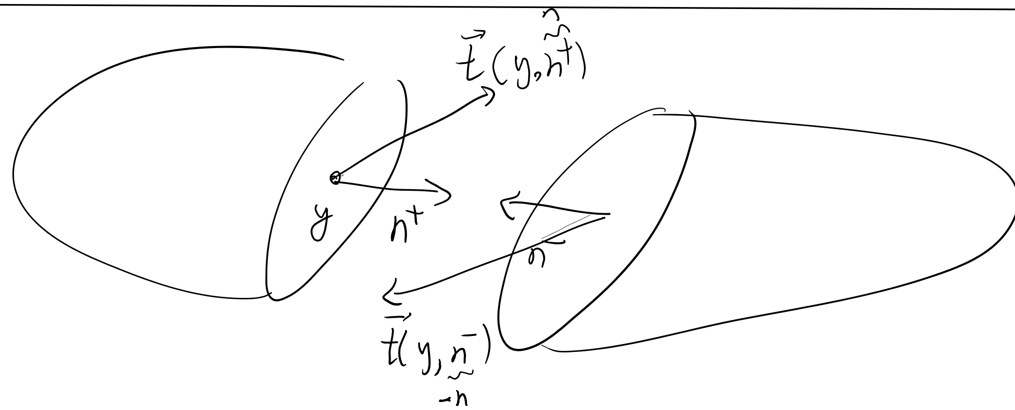
Traction and stress tensor:



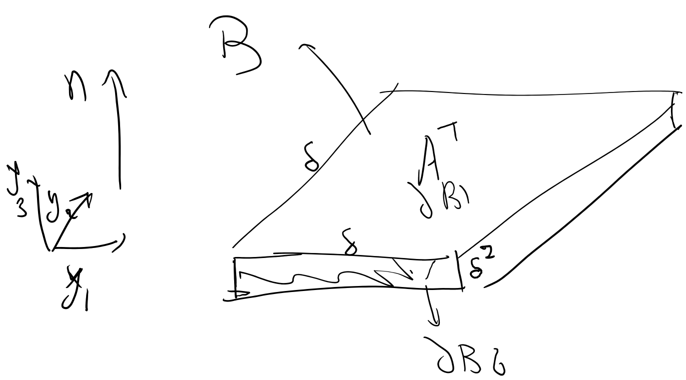
I mentioned that σ exists $\ni$

$\vec{t} = \sigma \, \vec{dS}$

why?



why $t(y,-n) = -t(y,n)$ Newton's action reaction law



$\delta \ll 1$

$V \approx \delta^4$

$A^+, A^- \sim \delta^2$

4 other surfaces on the side $\sim \delta^3$

w/o normal in it

Balance law

$D \int \rho v \, dV_n = \int \rho b \, dV_n + \int \vec{t} \, (d\mathcal{S})$

Balance ... in ...

$$\frac{D}{Dt}\int_B \rho v \, dV_y = \int_B \rho b \, dV_y + \int_{\partial B} \vec{t}_b \,(dS_y)$$

traction

Reduced transport eqn

$$\int \rho a \, dV_y = - \quad -$$

$\frac{Dv}{Dt}$ = acceleration

$$\underbrace{\int_B \rho(b - a)\, dV_y}_{\propto \delta^4} + \underbrace{\sum_{i=1}^{6}\int_{\partial B_i} \vec{t} \, dS_y}_{} = 0$$

all boundary integrals scale as δ^3 other than top & bottom

$\delta \to 0$

$$\Big(\underbrace{t(n)}_{\text{top}} + \underbrace{t(-n)}_{\text{bottom}}\Big)\delta^2 + O(\delta^3) = 0 \Rightarrow t(-n) = -t(n)$$

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3.4 The Cauchy Stress Tensor

We again consider a body $\overset{0}{\mathcal{B}}$ undergoing a motion $\{\mathbf{f}(\cdot,t)\}$ in response to a system of forces.

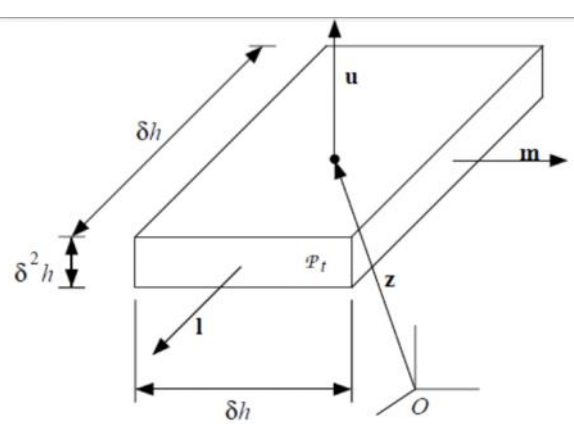


Figure 3.3: "Pillbox" for Cauchy's Action/Reaction Lemma

Theorem 154 (Cauchy's Action/Reaction Lemma) *For any given unit vector* $\mathbf{u}$,

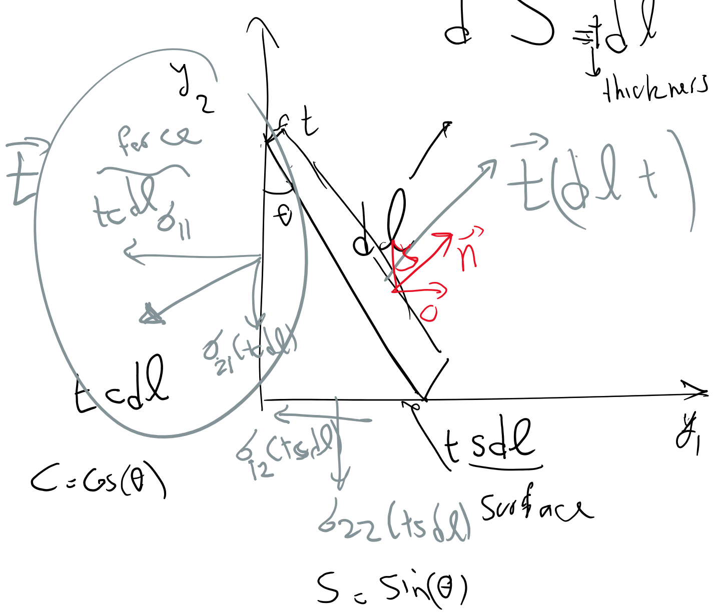
$$\mathbf{t}_{-\mathbf{u}}(\mathbf{y},t) = -\,\mathbf{t}_{\mathbf{u}}(\mathbf{y},t) \;\forall\; (\mathbf{y},t) \in \Im.$$

... h exists $\Rightarrow$ $t = hn$?

why σ exists $\ni$ $t = \sigma n$?

I'll do it in 2D

$dS = t\,dl$ thickness



normal to surface

$\sigma_{ij} =$

direction of traction

$$\underbrace{d\vec{F}}_{\text{force increment}} = \sigma\, \underbrace{d\vec{S}}_{\text{surface}}$$

$\Sigma F = 0$

balance of linear momentum

left face

$(-t c\,dl\ \sigma_{11})e_1 - (\sigma_{21} t c\,dl)\,e_2$

bottom surface $\leftarrow$ $(-\sigma_{12} t s\,dl)e_1 - (\sigma_{22} t s\,dl)e_2$

$\vec{t}\,dl\,t$

+ body force and acceleration inside $= 0$

$$\vec{t} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \underbrace{\begin{bmatrix} c \\ s \end{bmatrix}}_{n} = \boxed{\vec{t} = \sigma \cdot n}$$

for energy balance & thermal effects

flux of energy

$q_n = q \cdot n$

we can prove q exists

one can prove q exists

Please see this for the complete proof in 3D

Theorem 155 (Cauchy's Theorem on the Existence of the Stress Tensor) *$\exists$ a unique second-order tensor field $\mathbf{T}$ on the trajectory $\Im$, $\ni \forall$ unit vectors* $\mathbf{u}$,

$$\mathbf{t_u}(\mathbf{y},t) = \mathbf{T}(\mathbf{y},t)\mathbf{u} \ \forall \ (\mathbf{y},t) \in \Im.$$

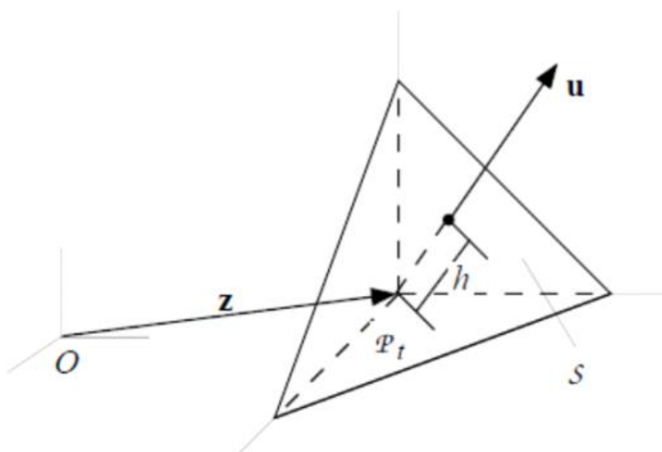
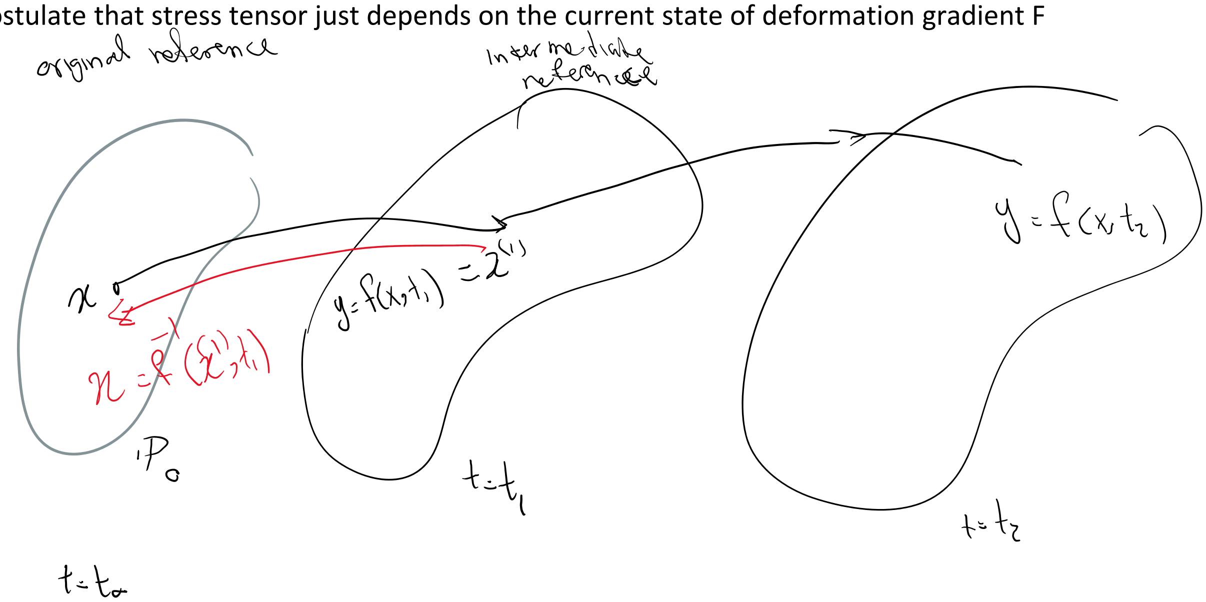


Figure 3.5: Tetrahedron of altitude h located at $\mathbf{z} \in \mathcal{B}_t$

Constitutive equations

Elastic material:
We postulate that stress tensor just depends on the current state of deformation gradient F



$t = t_2$

$t = t_o$

$$y = f(x, t)$$

$$\sigma = G(F, x)$$

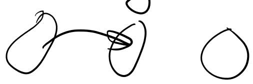
Cauchy stress

TAM551 notes

T

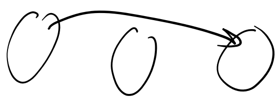
$$F = \nabla_x y = \nabla_x f(x,t)$$

ⓐ t_1 $\quad y = f(x, t_1) = \underbrace{x^{(1)}}_{\text{new reference}}$



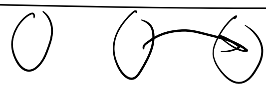
ⓘ

ⓐ t_2 $\quad y = f(x, t_2)$



ⓘⓘ

goal



$t_1 \quad t_2$

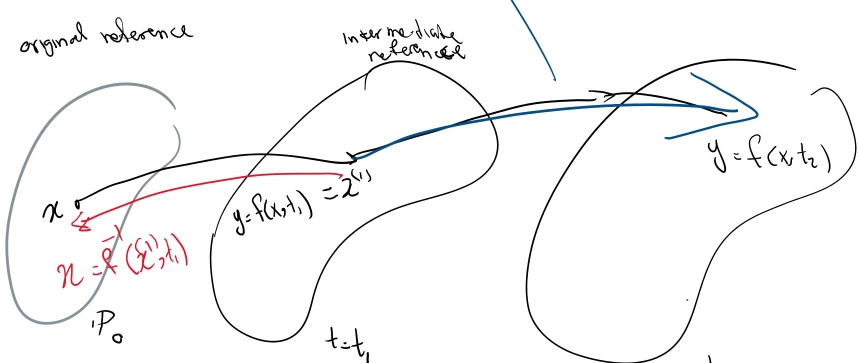
ⓘ $\quad x = f^{-1}(x^{(1)}, t_1)$

ii $\quad y = f(x, t_2) = f(f^{-1}(x^{(1)}, t_1), t_2)$

this is a map from new reference ⓐ t_1 to current configuration ⓐ t_2

$$y = f(f^{-1}(x^{(1)}, t_1), \underbrace{t_2 - t_1}_{\Delta t} + t_1)$$

$$y = f^{(2)}(x^{(1)}, t') \qquad t' = t_2 - t_1$$



$$y = f(x, t)$$

②

So, the goal is to relate a constitutive equation written with respect to one reference time to another one.

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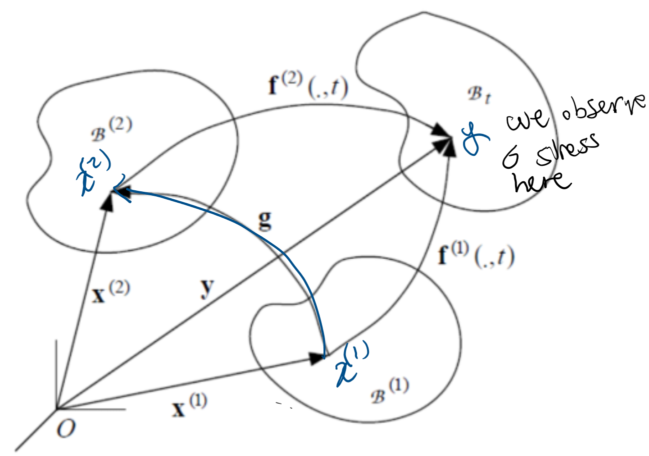


$\mathcal{B}^{(1)}$ to current configuration

$$\sigma(x^{(1)}) = G^{(1)}(F^{(1)}, x^{(1)})$$

$\mathcal{B}^{(2)}$ to current configuration

$$\sigma(x^{(2)}) = G^{(2)}(F^{(2)}, x^{(2)})$$

we want to relate $G^{(1)}$ & $G^{(2)}$

Figure 4.1: Alternative reference configurations for elastic response

$$\boxed{x^{(2)} = g(x^{(1)})} \quad \star$$ the map between two references

$$F^{(1)} = \nabla_{x^{(1)}} y \Rightarrow F^{(1)}_{ij} = \frac{\partial y_i}{\partial x^{(1)}_j} = \frac{\partial y_i}{\partial x^{(2)}_k}\frac{\partial x^{(2)}_k}{\partial x^{(1)}_j} = \frac{\partial y_i}{\partial x^{(2)}_k}\frac{\partial g(x^{(1)})_k}{\partial x^{(1)}_j}$$

$$= \frac{\partial y_i}{\partial x^{(2)}_k}\frac{\partial g_k}{\partial x_j} = F^{(2)}_{ik}(\nabla g)_{kj} \Longrightarrow$$

$$\boxed{F^{(1)} = F^{(2)}\nabla g}$$

$$\left.\begin{aligned} \sigma &= G^{(1)}(F^{(1)}, x^{(1)}) \\ &= G^{(2)}(F^{(2)}, x^{(2)}) \\ \text{but } F^{(1)} &= F^{(2)}\nabla g \end{aligned}\right\} \longrightarrow$$

$$G^{(2)}(F^{(2)}, x^{(2)}) = G^{(1)}(F^{(2)}\nabla g, x^{(1)})$$

$$G^{(2)}(F^{(2)}, x^{(2)}) = G^{(1)}(F \nabla g, x^{(1)})$$

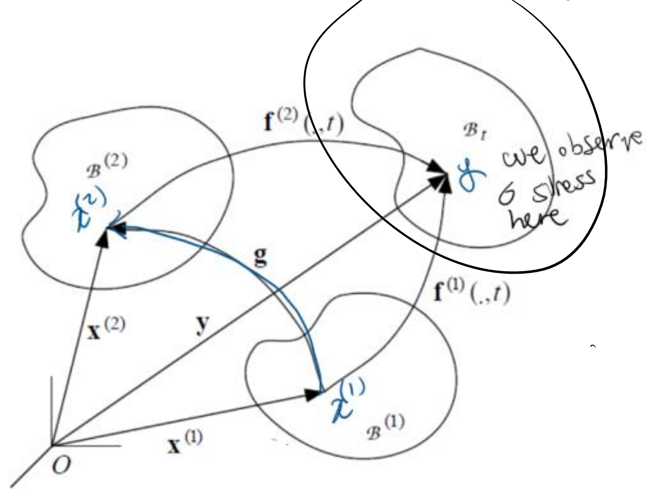
$x^{(1)} \to g^{-1}(x^{(2)})$ $\quad (x^{(2)} = g(x^{(1)}))$

So

$$\boxed{\forall F \quad G^{(2)}(F, x) = G^{(1)}(F \nabla g, g^{-1}(x))}$$

if we have G

the formula for $G^{(2)}$ can be obtained



(3)

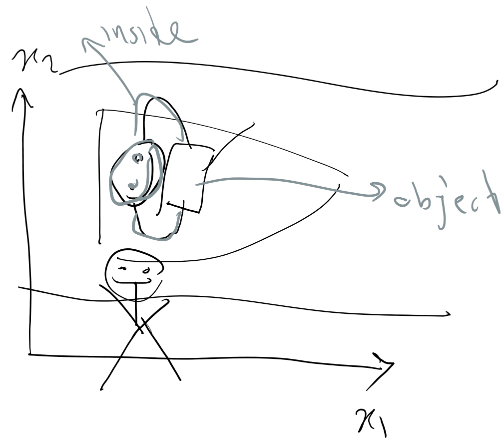
Now that we know how to relate constitutive equation from two different references, we can use objectivity to simplify the constitutive equation

4.3 Principle of Material Frame-Indifference

This section explores the notion that material response is invariant under (indifferent to) superposed rigid motions and shifts in the origin of the time scale. Only invariance under superposed rigid motion is relavent in the context of elasticity theory which does not include memory effects. We begin with the notion of equivalent motions.

Definition 110 *Two motions of a body, $\{\mathbf{f}(\cdot,t)\}$ and $\left\{\overset{*}{\mathbf{f}}(\cdot,t)\right\}$, are equivalent w.r.t. material response if they differ by a rigid deformation for each $t \in [t_0, \infty)$; i.e., $\exists$ functions $\mathbf{c} : [t_0,\infty) \to \mathcal{V}$ and $\mathbf{Q} : [t_0,\infty) \to \mathrm{Orth}\,\mathcal{V}^+ \ni$*

$$\overset{*}{\mathbf{f}}(\mathbf{x},t) = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{f}(\mathbf{x},t) \quad \forall\ (\mathbf{x},t) \in \overset{0}{\mathcal{B}} \times [t_0,\infty).$$



$y = f(x,t)$

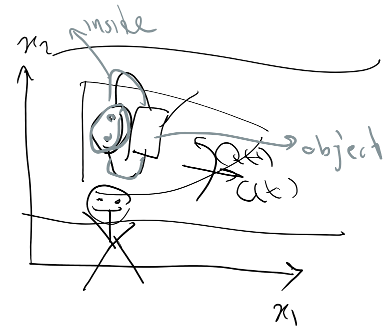
$$y = f(x,t)$$

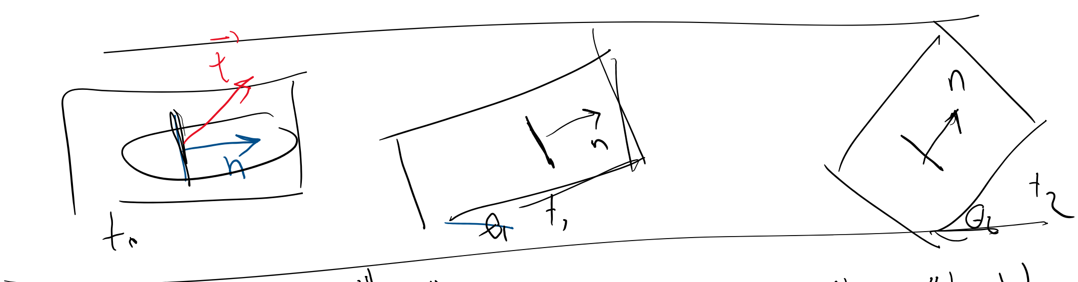
the boat has a rigid motion itself

the outside observer sees this deformation

$$y^*(x,t) = \underbrace{c(t)}_{\text{rigid translation}} + \underbrace{Q(t)}_{\text{rigid rotation}}\, y$$

$$\boxed{\underbrace{\vec{y}^*(x,t)}_{\text{outside observer}} = c(t) + Q(t)\,\underbrace{f(x,t)}_{\text{inside observer}}} \quad (4)$$





n rotates with the same angle as the "boat)

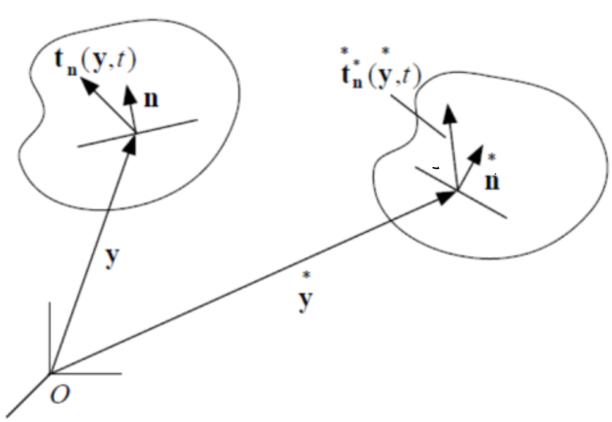


Figure 4.2: Surface tractions from equivalent motions

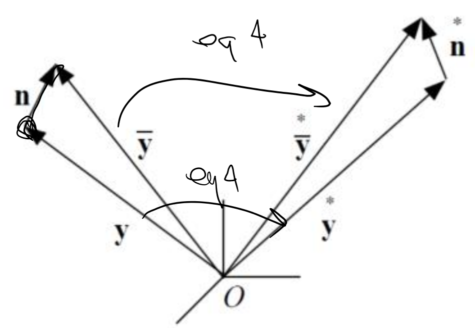


Figure 4.3: Relation between normals for equivalent motion

$\vec{y}$ & $\vec{n}$ given

I chose

I want to show that the * (outside) observer, sees the normal vectors rotated by Q(t)

I want to show that the * (outside) observer, sees the normal vectors rotated by Q(t)

I chose

$$\bar{y} = y + \vec{n}$$

$$\begin{cases} y^* = c(t) + Q(t)\, y \\ \bar{y}^* = c(t) + Q(t)\, \bar{y} \end{cases}$$

subtract

$$\bar{y}^* - y^* = Q(t)\,(\underbrace{\bar{y} - y}_{n}) \rightarrow \underbrace{\bar{y}^* - y^*}_{\text{orientation of } n} = Q(t)\, n$$

observed from (*) (outside)

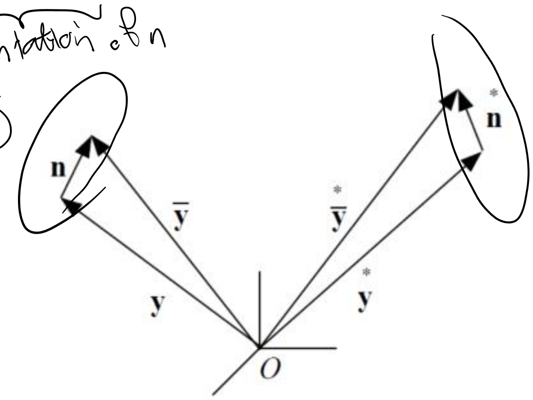


Figure 4.3: Relation between normals for equivalent motion

note

$Q(t)n$ is already of size 1:

$$(Qn)\cdot(Qn) = (\underbrace{Q^T Q}_{I}\, n)\cdot n = n \cdot n = 1$$

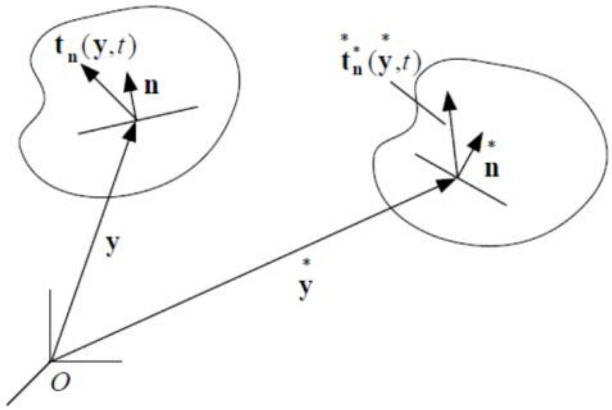
(5) $$\boxed{n^*(t) = Q(t)\, n}$$

Objectivity says

$$t^*(t) = Q(t)\, t$$

traction → time; time; traction

(6)



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