CM2023/12/04

Monday, December 4, 2023 9:40 AM

Last time we observed the relations below between two observers

$$y^*(x,t) = c(t) + Q(t)\, y(x,t)$$

$$\boxed{\begin{array}{ll} (1) & n^* = Q\, n \\ (2) & t_{n^*}^* = Q\, t_n \end{array}}$$

traction for normal vector n^*

$$(3)\quad t_{n^*}^* = \sigma^* n^*$$

$$(4)\quad t_n = \sigma n$$

premultiply by Q

(in TAM551 T is used for σ: Cauchy stress)

$$\underbrace{Q\, t_n}_{(2)\ =\ t_{n^*}^*} = Q \sigma n$$

$$\left.\begin{array}{l} t_{n^*}^* = Q\sigma n \\ (1)\quad n = Q^t n^* \end{array}\right\} \Longrightarrow \left.\begin{array}{l} t_{n^*}^* = (Q\sigma Q^t)\, n^* \\ (3)\quad t_{n^*}^* = \sigma^* n^* \end{array}\right\} \Longrightarrow$$

From objectivity

$$\boxed{\sigma^* = Q\sigma Q^t} \quad (5)$$

Now I want to recall the form of constitutive eqn from 2 different references?

$$y^*(x,t) = c(t) + Q(t)\, y(x,t)$$

$$F_{ij}^* = \frac{\partial y_i^*}{\partial x_j} = \frac{\partial\left(c_i(t) + Q_{im}(t)\, y_m(x,t)\right)}{\partial x_j} = Q_{im}(t)\, \frac{\partial y_m(x,t)}{\partial x_j}$$

$$= Q_{im} F_{mj}$$

6a) $F^* = QF$

6-b) $\sigma^* = G(F^*)$

6-b) $\sigma = G(F)$

5) $\sigma^* = Q \sigma Q^t$

$\Longrightarrow \underbrace{G(F^*)}_{\sigma^*} = Q \underbrace{G(F)}_{\sigma} Q^t \overset{6\text{-}a}{\Longrightarrow}$

$$\boxed{\forall Q \quad G(QF) = Q\, G(F)\, Q^t} \quad (7)$$

objectivity for elastic stress constitutive equation

Now, we use this objectivity equation to restrict the form of G:

Polar decomposition

$\underline{F = RU} \quad (= VR)$

$G(QF) = G(\underset{\text{any rotation that we can choose}}{Q}\underset{\text{rotation}}{R}U) = Q\, G(F)\, Q^t$

let's choose $Q = R^t$

$\Longrightarrow G(R^tRU) = G(U) = Q\,G(F)\,Q^t = R^t\, G(F)\, R$

$\longrightarrow \underline{G(F) = R\, G(U)\, R^t}$ 7a

$U = \sqrt{C}$

$C = F^tF$

$F = RU \rightarrow R = FU^{-1}$

$\rightarrow G(F) = FU^{-1}G(U)(FU^{-1})^t = F\underbrace{\left(U^{-1}G(U)U^{-t}\right)}_{\hat{G}(U)}F^t$

$\underline{G(F) = F\,\hat{G}(U)\,F^t}$ 7b

$= F\underbrace{\hat{G}(\sqrt{C})}_{\bar{G}(C)}F^t$ 7c

$$\overbrace{\bar{G}(C)}$$ 7c

Different Expressions of objective constitutive equations

$$\sigma = G(F) \quad \text{general elastic response}$$

$$+ \text{ Objectivity } \forall Q \quad G(QF) = Q\,G(F)Q^T \implies$$

Valid forms of G

$$G(F, x) = R(x)\, G(U, x)\, R^t_{(x)}$$
$$= F(x)\, \hat{G}(U, x)\, F^t(x)$$
$$= F(x)\, \bar{G}(C, x)\, F^t$$

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Theorem 173 *If the elastic constituitive equation*

$$\tilde{\mathbf{T}}(\mathbf{x}, t) = \mathbf{G}(\mathbf{F}(\mathbf{x}, t), \mathbf{x}) \tag{4.1}$$

is consistent with the Principle of Material Frame-Indifference, then it can be written in any of the following reduced forms:

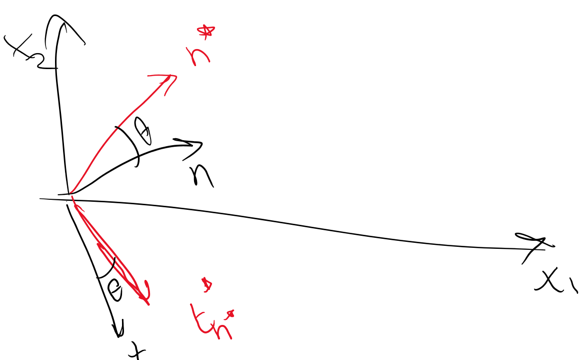
$$\tilde{\mathbf{T}}(\mathbf{x}, t) = \mathbf{R}(\mathbf{x}, t)\mathbf{G}(\mathbf{U}(\mathbf{x}, t), \mathbf{x})\mathbf{R}^t(\mathbf{x}, t); \tag{4.2}$$
$$\tilde{\mathbf{T}}(\mathbf{x}, t) = \mathbf{F}(\mathbf{x}, t)\hat{\mathbf{G}}(\mathbf{U}(\mathbf{x}, t), \mathbf{x})\mathbf{F}^t(\mathbf{x}, t); \tag{4.3}$$
$$\tilde{\mathbf{T}}(\mathbf{x}, t) = \mathbf{F}(\mathbf{x}, t)\bar{\mathbf{G}}(\mathbf{C}(\mathbf{x}, t), \mathbf{x})\mathbf{F}^t(\mathbf{x}, t); \tag{4.4}$$

where $\hat{\mathbf{G}} : \mathrm{Psym} \times \overset{0}{\mathcal{B}} \to \mathrm{Sym}$ *and* $\bar{\mathbf{G}} : \mathrm{Psym} \times \overset{0}{\mathcal{B}} \to \mathrm{Sym}$.

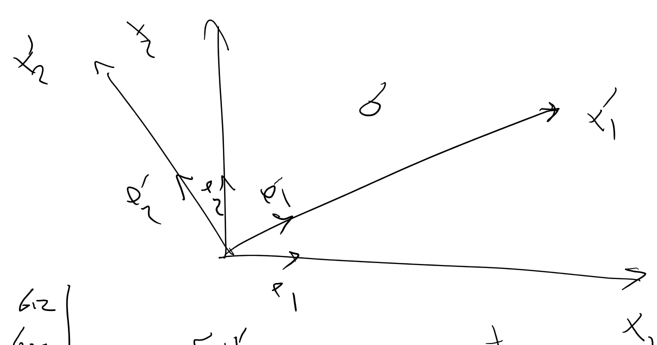
Conversely, an elastic response function written in any of these reduced forms is consistent with the Principle of Material Frame-Indifference $\forall$ *choices of* $\mathbf{G}$, $\hat{\mathbf{G}}$ *and* $\bar{\mathbf{G}}$.

$$\boxed{\sigma^* = Q \sigma Q^t}$$ 5

and how this equation is related to coordinate transformation



$n^* = Qn$



$$[\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

$$[\sigma]' = Q\,[\sigma]\,Q^t$$

$$Q = \begin{bmatrix} e_1' \\ \hline e_2' \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

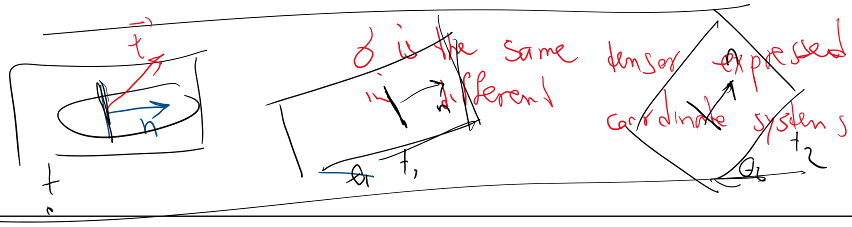
t_n

$n^* \neq n$ $\quad n^* = Qn$

$t_n^* \neq t_n$ $\quad t_n^* = Qt_n$ $\quad \sigma^* = Q\sigma Q^T$

$\sigma^* \neq \sigma$

$Q = \begin{bmatrix} e_1' \\ e_2' \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$



specific energy

$$e^*(F^*) = e(F)$$

$y^* = c + Qy$

$$T^*_{ijk\ldots l} = Q_{ii'}Q_{jj'}\cdots Q_{ll'}T_{i'\ldots l'} \qquad (8)$$

Objectivity for any tensor

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objective, taking the material time derivative of $\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^T$ leads to $\dot{\mathbf{T}}^* = \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^T + \dot{\mathbf{Q}}\mathbf{T}\mathbf{Q}^T + \mathbf{Q}\mathbf{T}\dot{\mathbf{Q}}^T$ and so $\dot{\mathbf{T}}$ is not objective (except under Gallilean transformations where $\mathbf{Q}$ is time independent). However, as was shown in one of the problems in Section 3.6, the co-rotational derivative of $\mathbf{T}$ defined by

$$\overset{\triangle}{\mathbf{T}} = \dot{\mathbf{T}} + \mathbf{L}^T\mathbf{T} + \mathbf{T}\mathbf{L},$$

is objective. All of the following quantities, each of which has the dimension of stress rate, can be shown to be objective:

$$\begin{aligned}
\overset{\triangle}{\mathbf{T}} &= \dot{\mathbf{T}} + \mathbf{L}^T\mathbf{T} + \mathbf{T}\mathbf{L} && \text{Convected rate,}\\
\overset{\nabla}{\mathbf{T}} &= \dot{\mathbf{T}} - \mathbf{L}\mathbf{T} - \mathbf{T}\mathbf{L}^T && \text{Oldroyd rate,}\\
\overset{\circ}{\mathbf{T}} &= \dot{\mathbf{T}} - \mathbf{W}\mathbf{T} + \mathbf{T}\mathbf{W} && \text{Co-rotational or Jaumann rate,}\\
\overset{\otimes}{\mathbf{T}} &= \dot{\mathbf{T}} + \mathbf{T}\boldsymbol{\Omega} - \boldsymbol{\Omega}\mathbf{T} && \text{Green-Naghdi rate,}\\
\overset{\square}{\mathbf{T}} &= \tfrac{1}{2}(\overset{\triangle}{\mathbf{T}} - \overset{\nabla}{\mathbf{T}}) = \mathbf{D}\mathbf{T} + \mathbf{T}\mathbf{D};
\end{aligned}$$

These are objective rates of stress
(finite strain dynamic or hypoelastic materials)

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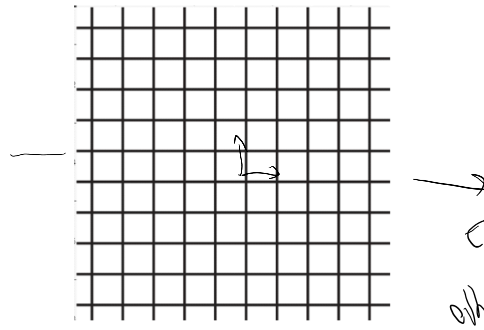
4.4 material symmetry; Isotropy

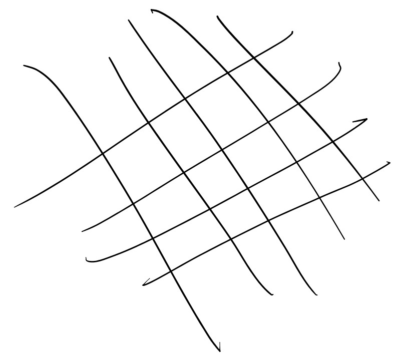
$$\sigma = C(\theta)\varepsilon$$

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \qquad \text{4th order} \qquad \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{bmatrix}$$



$\varepsilon_{11} = 1$

other esp = 0

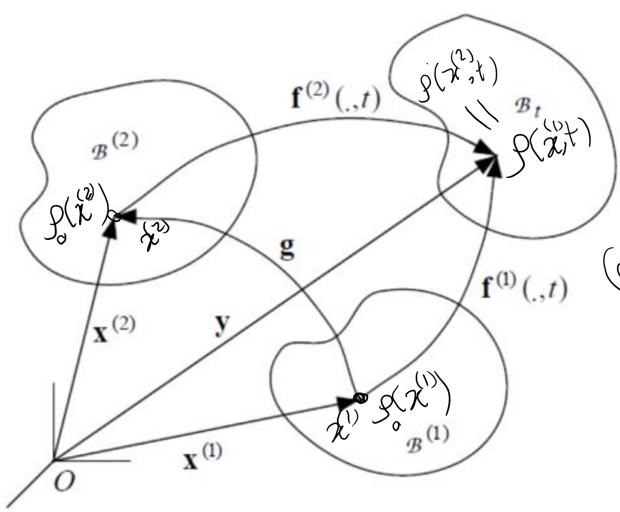


$C(30^\circ)$

$$C(30^\circ) \neq C(0^\circ)$$

$$C(90^\circ) = C(0^\circ)$$

Any multiple of 90 degree rotation here does not change the geometry => the constitutive equation does not change



Lagrangian balance of mass

Ref ① → current

$$\text{(9a)} \begin{cases} \dfrac{\rho(x^{(1)},t)}{\rho_0(x^{(1)})} = J_1 = \det F^{(1)} \\ F^{(1)} = \nabla_{x^{(1)}} f^{(1)} \end{cases}$$

Ref ② → current

$$\text{(9b)} \begin{cases} \dfrac{\rho(x^{(2)},t)}{\rho_0(x^{(2)})} = J_2 = \det F^{(2)} \\ F^{(2)} = \nabla_{x^{(2)}} f^{(2)} \end{cases}$$

... (9a) / (9b)

$$\frac{\rho(x^{(1)},t)}{\rho(x^{(2)},t)} \cdot \frac{\rho_0(x^{(2)})}{\rho_0(x^{(1)})} = \frac{\det F^{(1)}}{\det F^{(2)}}$$

(9c) Note that $\rho(x^{(1)},t) = \rho(x^{(2)},t)$

$$\frac{\rho(x^{(1)},t)}{\rho(x^{(2)},t)} \cdot \frac{\rho_o(x^{(2)})}{\rho_o(x^{(1)})} = \frac{\det F^{(1)}}{\det F^{(2)}}$$

$= 1$ from (9c)

$$L \quad F^{(2)} \cdot \nabla_{x^{(2)}} \rho^{(2)}$$

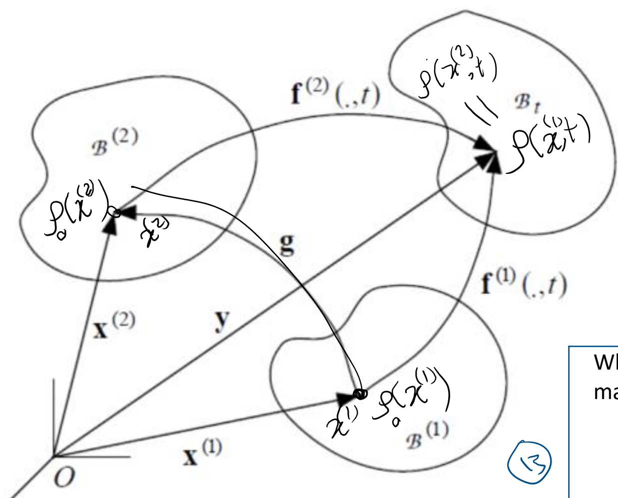
(9c) Note that $\rho(x^{(1)};t) = \rho(x^{(2)},t)$

$$\boxed{\rho(x^{(1)},t) = \rho(x^{(2)},t)\ \frac{\det F^{(1)}}{\det F^{(2)}}} \qquad (10)$$

$$x^{(2)} = g(x^{(1)}) \Longrightarrow \quad F^{(1)}_{ij} = \frac{\partial f_i}{\partial x^{(1)}_j} = \frac{\partial f_i}{\partial x^{(2)}_k}\frac{\partial x^{(2)}_k}{\partial x^{(1)}_j} = F^{(2)}_{ik}\frac{\partial g_k(x^{(1)})}{\partial x^{(1)}_j} = F^{(2)}_{ik}(\nabla g)_{kj}$$

$$\boxed{F^{(1)} = F^{(2)}\nabla g} \qquad (11)$$

$$(10)\ \&\ (11) \Longrightarrow \quad \rho(x^{(1)},t) = \rho(x^{(2)},t)\ \frac{\det F^{(2)}\nabla g}{\det F^{(2)}} = \rho(x^{(2)},t)\det\nabla g$$



(13)

$$\boxed{\rho(x^{(1)},t) = \rho(x^{(2)},t)\det\nabla g} \qquad (12)$$

(1) density — (2) density — Jacobian between the two coordinate system

What is the condition that the two references have the same initial density for material?

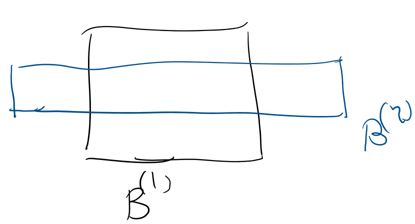
$$\det \nabla g = 1$$

A second order tensor whose determinant is 1 is called a unimodal tensor

$H \qquad \det H = 1$

Examples of maps that preserve density:

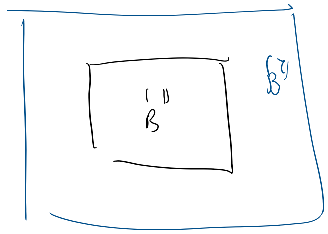
$$\nabla g = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$



$\det \nabla g = 1$

$\rho_o^{(1)} = \rho_o^{(2)}$ good :)

$$\nabla g = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$



$$\rho^{(1)} = \rho^{(2)} \det \nabla g = \underline{\rho^{(2)} 4}$$

Inviscid fluid only have pressure and pressure does not change as long as we don't change the density.
This is the rationale, why in determining the symmetries of the stress constitutive equation, we allow unimodal transfers between two reference coordinates, because for inviscid fluids this is in fact the right space

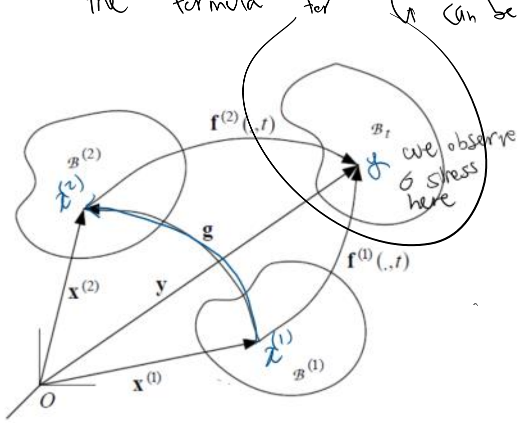
$$\mathrm{Unim}\, V^{+} = \{ H \in \underbrace{\mathrm{Lin}\, V}_{\text{second order}} \mid \det H = 1 \}$$

Last time we observed

$$\forall F \qquad G^{(2)}(F, x) = G^{(1)}(F\nabla g, g^{-1}(x))$$

if we have G

the formula for $G^{(2)}$ can be obtained



(3)
last time eqn (3)

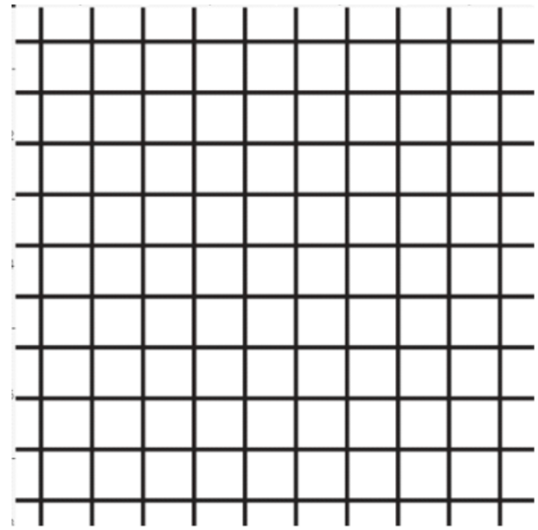
We limit ourselves to g's such that det(grad g) = 1 (unimodal transformations)

Example $\quad g = R(\theta)$

$$G^{(2)}(F) = G^{(1)}(F \nabla g)$$

what if $\quad \theta = n \frac{\pi}{2}$

$$G^{(1)} = G^{(2)}$$



in general if ∇g is in a symmetry group

$$G^{(2)} = G^{(1)}$$

$$G(F) = G(F \nabla g)$$

∇g belongs to the symmetry group for stress constitutive eqn

Definition 112 (Noll, 1958) *Given an elastic body and a reference configuration that corresponds to the region* $\overset{0}{\mathcal{B}}$, *the* material symmetry group *at the material point identified by* $\mathbf{x}$ *in the reference configuration is the set*

$$\underset{\mathbf{x}}{\mathrm{Msg}} = \left\{ \underset{\nabla g}{\mathbf{H}} \in \mathrm{Unim}\ \mathcal{V}^{+} : \mathbf{G}(\mathbf{FH}, \mathbf{x}) = \mathbf{G}(\mathbf{F}, \mathbf{x})\ \forall\ \mathbf{F} \in \mathrm{Lin}\ \mathcal{V}^{+} \right\}.$$

Again, it should be emphasized that the material symmetry group is characterized by tensors $\mathbf{H}$ that correspond to the gradients at $\mathbf{x}$ of deformations — not the deformations themselves. This is because the mass density and the elastic response function in the second reference configuration depend only on the gradient of the connecting deformation. Also, note that $\mathbf{H} \in \mathrm{Msg}_{\mathbf{x}}$ is not a tensor field, but rather the value of a tensor field at $\mathbf{x}$.

The following theorem presents a property of all orthogonal elements of $\mathrm{Msg}_{\mathbf{x}}$ that derives from the Principle of Material Frame-Indifference.