From last time:

Definition 112 (Noll, 1958) *Given an elastic body and a reference configuration that corresponds to the region* $\overset{0}{\mathcal{B}}$, *the* material symmetry group *at the material point identified by* $\mathbf{x}$ *in the reference configuration is the set*

$$\underset{\mathbf{x}}{\mathrm{Msg}} = \left\{ \mathbf{H} \in \mathrm{Unim}\ \mathcal{V}^+ : \mathbf{G}(\mathbf{FH}, \mathbf{x}) = \mathbf{G}(\mathbf{F}, \mathbf{x})\ \forall\ \mathbf{F} \in \mathrm{Lin}\ \mathcal{V}^+ \right\}.$$

Again, it should be emphasized that the material symmetry group is characterized by tensors $\mathbf{H}$ that correspond to the gradients at $\mathbf{x}$ of deformations — not the deformations themselves. This is because the mass density and the elastic response function in the second reference configuration depend only on the gradient of the connecting deformation. Also, note that $\mathbf{H} \in \mathrm{Msg}_{\mathbf{x}}$ is not a tensor field, but rather the value of a tensor field at $\mathbf{x}$.

The following theorem presents a property of all orthogonal elements of $\mathrm{Msg}_{\mathbf{x}}$ that derives from the Principle of Material Frame-Indifference.

Theorem 174 *Let* $\mathrm{Q} \in \mathrm{Orth}\ \mathcal{V}^+$. *Then* $\mathrm{Q} \in \mathrm{Msg}_{\mathbf{x}}$ *iff*

$$\mathrm{G}(\mathrm{QFQ}^t, \mathrm{x}) = \mathrm{QG}(\mathrm{F}, \mathrm{x})\mathrm{Q}^t\ \forall\ \mathrm{F} \in \mathrm{Lin}\ \mathcal{V}^+.$$

If Q is a rotation, for it to belong to symmetry group of constitutive equation G, we must have

$$G(QFQ^t, x) = Q\ G(F, x)Q^t \quad \forall F \in \mathrm{Lin}\,V^+ \quad (\det F > 0)$$

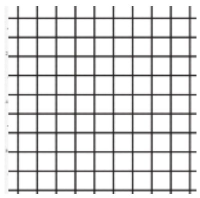
$$\boxed{\begin{array}{llll} Q \in \mathrm{Msg}_x \Longrightarrow & (1) & G(F'\underset{Q \text{ on } F}{\underline{Q}}) = G(F') & \text{Noll's definition} \\ \text{objectivity} & (2) & G(\underset{Q \text{ on } L}{\underline{Q}}F') = Q\ G(F)\ Q^t & \end{array}}$$

$$F' = QFQ^t \overset{(1)}{\Longrightarrow} G((QF\underbrace{Q^t)Q}_{I}) = G(QFQ^t) \Longrightarrow$$

$$G(QFQ^t) = G(QF) \qquad \text{use eqn (2)}$$

$$= Q\ G(F)\ Q^t$$

$$G(QFQ^t) = Q\,G(F)\,Q^t \quad \text{if } Q \in M_{sg}$$



rotation by $90^\circ n$

Q

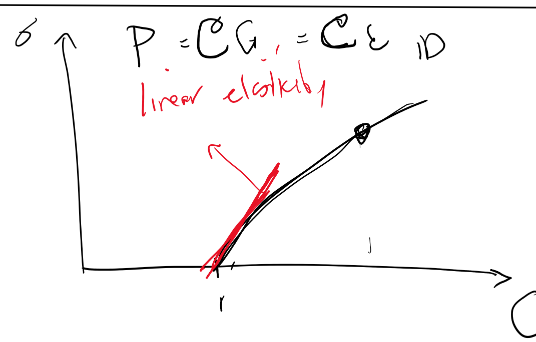
$Q = Rot(n60^\circ)$

I will comment on how this can be simplified for linear elasticity (relevant to HW9)

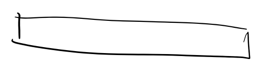
Linear Elasticity:

$$\sigma = F\bar{G}(C)F^t$$

$C = F^tF\,,\quad F = \nabla y_{/x}$



4th order elasticity tensor



$$\left.\begin{aligned}\sigma &= F\bar{G}(C)F^t \\ P &= J\sigma F^{-t}\end{aligned}\right\} \rightarrow P = J(F\bar{G}(C)F^t)F^{-t} \Rightarrow$$

$P \downarrow$ PK-I

$$\boxed{P = JF\bar{G}(C)} \quad (1)$$

we'll do a Taylor's series around $C = I$ for $\bar{G}(C)$

$$f(x+1) = f(1) + x f'(1) + \frac{x^2}{2} f''(1) + \dots \quad \left(\frac{\partial^2 f}{\partial x^2}\right)$$

$$P = JF\bar{G}(C) = JF(\bar{G}(C - I + I)) = JF\Big(\underbrace{\bar{G}(I)}_{0} + (C-I)\left(\frac{\partial \bar{G}(I)}{\partial C}\right) + O((C-I)^2)\Big)$$

we assume that the reference is such that for $F = I$ $(C = I)$ we have zero stress

Constitutive eqn

$$P = JF\left[\left(2\frac{\partial \bar{G}(I)}{\partial C}\right)\left(\frac{C - I}{2}\right) + O(\varepsilon^2)\right] = JF\,ℂ\,\varepsilon$$

$$\underbrace{\quad}_{\substack{\text{4th order}\\ \text{tensor}}} \qquad \underbrace{\quad}_{G} \quad + O(\varepsilon) \qquad \varepsilon = \|H\| \qquad \text{Green strain}$$

$$\mathbb{C}_{ijkl} = 2\frac{\partial \bar{G}_{ij}}{\partial C_{kl}}(I)$$

$$P = \underbrace{\left(1 + \overbrace{\operatorname{trace}(E)}^{O(\varepsilon)} + O(\varepsilon^2)\right)}_{\substack{J\\ \text{HW7}}} \overbrace{\left(I + \underset{O(\varepsilon)}{H}\right)}^{F}\left(\mathbb{C}G + O(\varepsilon^2)\right)$$

$$= \mathbb{C}G + O(\varepsilon)$$

$$\boxed{P = \mathbb{C}G + O(\varepsilon)} \qquad \varepsilon \to \|H\|$$

strain $\left(\frac{\nabla u + \nabla u^T}{2}\right)$

noting $G = \varepsilon + O(\varepsilon^2)$ $\quad \|H\| = \varepsilon$

σ, P, S the same within $O(\varepsilon^2)$?

linear elasticity

$$\mathbb{C} = 2\frac{\partial \bar{G}}{\partial C}(I)$$

$$\begin{matrix} P \\ S \\ \sigma \end{matrix} = \mathbb{C} \begin{matrix} \varepsilon \\ G \end{matrix}$$

$\sigma = \mathbb{C}\varepsilon$ for linear elasticity

$$\sigma_{ij} = \mathbb{C}_{ijkl}\,\varepsilon_{lk}$$

2

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \\ & & \\ & & \sigma_{33} \end{bmatrix}$$ same with ε in 3D

9 comp $\quad$ 3×3

What about $\mathbb{C}$ (3×3) $\quad 3\times3\times3\times3 = 81$ components.

$$\sigma_{ij} = \sigma_{ji} \longrightarrow \mathbb{C}_{ijkl} = \mathbb{C}_{jikl}$$

$$\mathbb{C}_{ijkl} = 2\,\frac{\partial \bar{G}_{ij}(C)}{\partial C_{kl}} = 2\,\frac{\partial \bar{G}_{ij}(C)}{\partial C_{lk}} = \mathbb{C}_{ijlk}$$

because $C_{kl} = C_{lk}$ $(C = C^T)$

$$\boxed{\begin{array}{l}\text{minor symmetries of } \mathbb{C} \text{ tensor} \\ \mathbb{C}_{ijkl} = \mathbb{C}_{jikl} \\ \mathbb{C}_{ijkl} = \mathbb{C}_{ijlk}\end{array}}$$

3

Voigt notation:

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}$$

independent values

σ is sym

$$\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix}$$

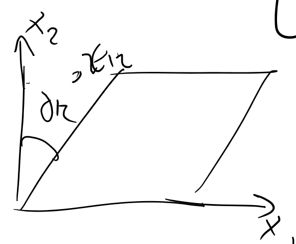
independent values

$$S = \begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \hline \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} \qquad \begin{matrix}\sigma_{12}\\ \sigma_{23}\\ \sigma_{31}=\sigma_{13}\end{matrix}$$

Voigt stress array

$$\gamma = \begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{33}\\ \hline 2\epsilon_{23}\\ 2\epsilon_{13}\\ 2\epsilon_{12}\end{bmatrix} = \begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{33}\\ \hline \gamma_{23}\\ \gamma_{13}\\ \gamma_{12}\end{bmatrix}$$

eng shear strains



$$S = \underbrace{\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix}}_{6\times1} = \underbrace{S}_{6\times 6}\ \underbrace{\gamma}_{6\times 1}$$

S → Voigt stiffness

$$C_{\underset{ji}{ij}\,kl}$$

$ij = ji$ → 6 indep comb

kl → 6 indep. combinati

out of 81 values in C_{ijkl} only 36 are independent

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \vdots\end{bmatrix} = \left[\begin{array}{c|c} \underset{3\times3}{\epsilon_n \to \sigma_n} & \underset{3\times3}{\epsilon_{shear} \to \sigma_n} \\ \hline \epsilon_n \to \sigma_{shear} & \epsilon_{shears} \to \sigma_{shear} \end{array}\right] \begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{33}\\ 2\epsilon_{23}\\ 2\epsilon_{13}\\ \vdots\end{bmatrix}$$

4

$$\begin{pmatrix} \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} \quad \left[\begin{array}{c|c} \sigma_n \to \sigma_{shear} & \epsilon_{shears} \to \sigma_{shear} \\ 3\times 3 & 3\times 3 \end{array} \right] \begin{pmatrix} 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{pmatrix} \quad (4)$$

Voigt notation

Hyperelastic materials

General elastic material $\sigma = G(F) \overset{\text{objectivity}}{=} F\,\bar{G}(C)\,F^t$

Hyperelastic material is more specialized $e = \bar{e}(F) \overset{\text{objectivity}}{=} \bar{e}(C)$

$e \downarrow$ specific energy

$$\Longrightarrow \quad \bar{G}(C) = \frac{2\rho_0}{\sqrt{\det C}} \frac{\partial \bar{e}(C)}{\partial C} \quad (5)$$

for the proof see Hyperelastic.pdf

What's the formula for elasticity tensor $\mathbb{C}$ for a hyperelastic material

$$\mathbb{C} = 2 \frac{\partial \bar{G}}{\partial C}(I)$$

we derived this above for any elastic material $\Big\}\Longrightarrow$

hyper elastic $\bar{G}(C) = \frac{2\rho_0}{\sqrt{\det C}} \frac{\partial \bar{e}(C)}{\partial C}$

$$\mathbb{C} = 2 \frac{\partial \left(\underbrace{\frac{2\rho_0}{\sqrt{\det C}}}_{a} \underbrace{\left(\frac{\partial \bar{e}(C)}{\partial C}\right)}_{b} \right)}{} \Bigg| =$$

$$\mathcal{C}^{\gamma} = 2 \frac{\partial \left(\frac{2\rho_0}{\sqrt{\det C}} \left(\frac{\partial \bar{e}}{\partial C} \right) \right)}{\partial C} \Bigg|_{C=I} =$$

$$4\rho_0 \left(\frac{\partial \frac{1}{\sqrt{\det C}}}{\partial C} \frac{\partial \bar{e}(C)}{\partial C} + \frac{1}{\sqrt{\det C}} \frac{\partial^2 \bar{e}(C)}{\partial C \partial C} \right) \Bigg|_{C=I}$$

$(ab)' = a'b + ab'$

zero

$$= 4\rho_0 \left[\frac{\partial \frac{1}{\sqrt{\det C}}}{\partial C} \Bigg|_{C=I} \frac{\partial \bar{e}}{\partial C}(I) + \frac{1}{\sqrt{\det I}} \frac{\partial^2 \bar{e}}{\partial C \partial C}(I) \right]$$

Recall $\gamma = F \bar{G}(C) F^t = F \frac{2\rho_0}{\sqrt{\det C}} \frac{\partial \bar{e}}{\partial C}(C) F^t$

if $C = I$ $\quad \gamma(C=I) = F \frac{2\rho_0}{\sqrt{1}} \frac{\partial \bar{e}}{\partial C}(I) F^t = 0$

zero

For hyperelastic materials we have

$$\mathcal{C}^{\gamma} = 4\rho_0 \frac{\partial^2 \bar{e}}{\partial C \partial C}(C=I)$$

$$\mathcal{C}^{\gamma}_{ijkl} = 4\rho_0 \frac{\partial^2 \bar{e}}{\partial C_{ij} \partial C_{kl}}(C=I)$$

$$= 4\rho_0 \frac{\partial^2 \bar{e}}{\partial C_{kl} \partial C_{ij}}(C=I) = \mathcal{C}_{klij}$$

$$\boxed{\frac{\partial^2 f}{\partial a \partial b} = \frac{\partial^2 f}{\partial b \partial a}}$$

$$\frac{\partial^2 \psi}{\partial a \partial b} = \frac{\partial^2 \psi}{\partial b \partial a}$$

For hyperelastic

$$\underset{\sim}{C} = 4\rho_0 \frac{\partial^2 \bar{e}}{\partial C \partial C}(I)$$

$$\underset{\sim}{C}_{\tilde{i}j\,kl} = C_{kl\,\tilde{i}j} \quad \text{major symmetry}$$

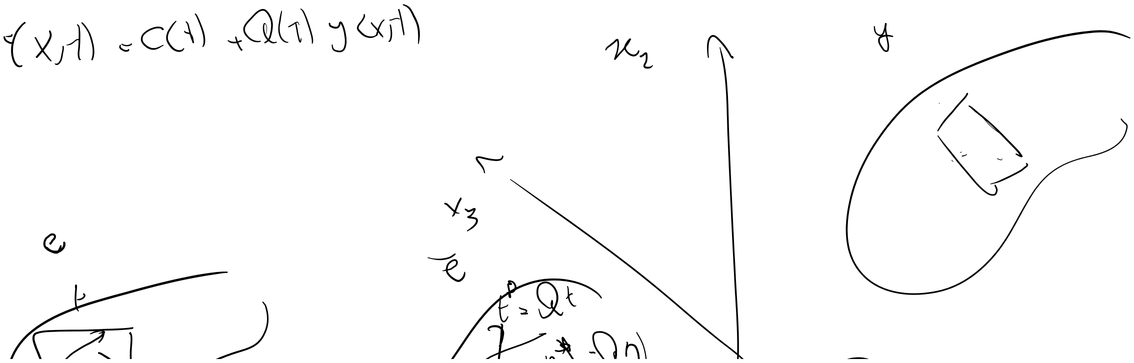
& it still has minor symmetries that any elastic material has

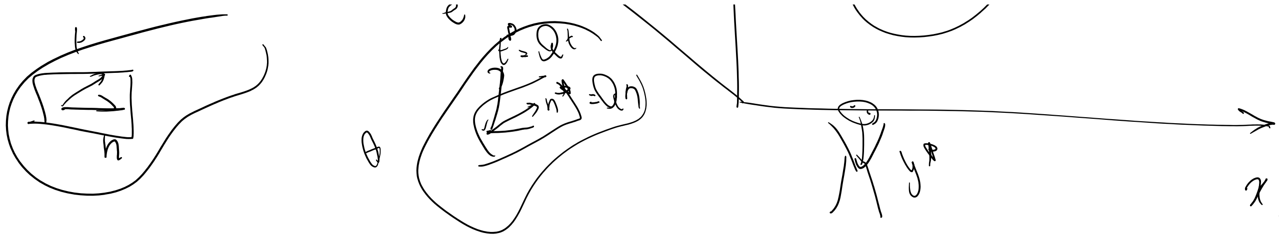
$$C_{\tilde{i}jkl} = C_{jikl} \quad , \quad C_{ijkl} = C_{ijlk}$$

$s = S\gamma$ in this case S is symmetric for hyperelastic materic

$$s = \begin{array}{c} 21 \ \#s \\ \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ & S_{22} & & & & \\ & & \ddots & & & \\ & \text{sym} & & & & \\ & & & & & S_{66} \end{bmatrix} \end{array} \gamma$$

$$y^*(X,t) = c(t) + Q(t)\, y(X,t)$$



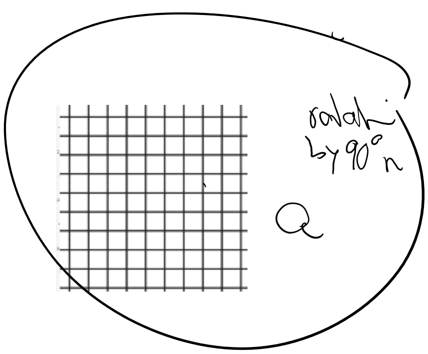


$$\sigma^* = Q \sigma Q^T \qquad \sigma^*_{ij} = Q_{im} Q_{jn} \sigma_{mn}$$

$$\mathbb{C}^* = ?$$

$$\mathbb{C}^*_{ijkl} = Q_{im} Q_{jn} Q_{kp} Q_{lq} \mathbb{C}_{ijkl}$$

What if Q is in symmetry group for constitutive equation of stress ($\mathbb{C}$)



$\supset Q = Rot(n60°)$

Q is in symmetry group iff

for $$\mathbb{C}^*_{ijkl} = Q_{im} Q_{jn} Q_{kp} Q_{lq} \mathbb{C}_{mnpq}$$

$$\mathbb{C}^*_{ijkl} = \mathbb{C}_{ijkl}$$

For an isotropic elastic material, we only have 2 independent parameters

E, ν

E, ν

λ, μ : Lamé's parameter

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

https://en.wikipedia.org/wiki/Lam%C3%A9_parameters

3D formulae	$K =$	$E =$	$\lambda =$	$G =$	$\nu =$	$M =$	Notes
$(K,\, E)$			$\frac{3K(3K-E)}{9K-E}$	$\frac{3KE}{9K-E}$	$\frac{3K-E}{6K}$	$\frac{3K(3K+E)}{9K-E}$	
$(K,\, \lambda)$		$\frac{9K(K-\lambda)}{3K-\lambda}$		$\frac{3(K-\lambda)}{2}$	$\frac{\lambda}{3K-\lambda}$	$3K - 2\lambda$	
$(K,\, G)$		$\frac{9KG}{3K+G}$	$K - \frac{2G}{3}$		$\frac{3K-2G}{2(3K+G)}$	$K + \frac{4G}{3}$	
$(K,\, \nu)$		$3K(1-2\nu)$	$\frac{3K\nu}{1+\nu}$	$\frac{3K(1-2\nu)}{2(1+\nu)}$		$\frac{3K(1-\nu)}{1+\nu}$	
$(K,\, M)$		$\frac{9K(M-K)}{3K+M}$	$\frac{3K-M}{2}$	$\frac{3(M-K)}{4}$	$\frac{3K-M}{3K+M}$		
$(E,\, \lambda)$	$\frac{E+3\lambda+R}{6}$			$\frac{E-3\lambda+R}{4}$	$\frac{2\lambda}{E+\lambda+R}$	$\frac{E-\lambda+R}{2}$	$R = \sqrt{E^2 + 9\lambda^2 + 2E\lambda}$
$(E,\, G)$	$\frac{EG}{3(3G-E)}$		$\frac{G(E-2G)}{3G-E}$		$\frac{E}{2G} - 1$	$\frac{G(4G-E)}{3G-E}$	
$(E,\, \nu)$	$\frac{E}{3(1-2\nu)}$		$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{E}{2(1+\nu)}$		$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$	
$(E,\, M)$	$\frac{3M-E+S}{6}$		$\frac{M-E+S}{4}$	$\frac{3M+E-S}{8}$	$\frac{E-M+S}{4M}$		$S = \pm\sqrt{E^2 + 9M^2 - 10EM}$ There are two valid solutions. The plus sign leads to $\nu \geq 0$. The minus sign leads to $\nu \leq 0$.
$(\lambda,\, G)$	$\lambda + \frac{2G}{3}$	$\frac{G(3\lambda+2G)}{\lambda+G}$			$\frac{\lambda}{2(\lambda+G)}$	$\lambda + 2G$	
$(\lambda,\, \nu)$	$\frac{\lambda(1+\nu)}{3\nu}$	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$		$\frac{\lambda(1-2\nu)}{2\nu}$		$\frac{\lambda(1-\nu)}{\nu}$	Cannot be used when $\nu = 0 \Leftrightarrow \lambda = 0$
$(\lambda,\, M)$	$\frac{M+2\lambda}{3}$	$\frac{(M-\lambda)(M+2\lambda)}{M+\lambda}$		$\frac{M-\lambda}{2}$	$\frac{\lambda}{M+\lambda}$		
$(G,\, \nu)$	$\frac{2G(1+\nu)}{3(1-2\nu)}$	$2G(1+\nu)$	$\frac{2G\nu}{1-2\nu}$			$\frac{2G(1-\nu)}{1-2\nu}$	
$(G,\, M)$	$M - \frac{4G}{3}$	$\frac{G(3M-4G)}{M-G}$	$M - 2G$		$\frac{M-2G}{2M-2G}$		
$(\nu,\, M)$	$\frac{M(1+\nu)}{3(1-\nu)}$	$\frac{M(1+\nu)(1-2\nu)}{1-\nu}$	$\frac{M\nu}{1-\nu}$	$\frac{M(1-2\nu)}{2(1-\nu)}$			
2D formulae	$K_{2D} =$	$E_{2D} =$	$\lambda_{2D} =$	$G_{2D} =$	$\nu_{2D} =$	$M_{2D} =$	Notes

...