1. Figure 1 shows a possible jump manifold Γ in a domain $\mathcal{D}$. The balance law,

$$\int_{\partial\Omega} \mathbf{F}.\mathbf{n}\,\mathrm{dS} + \int_{\Omega} \mathbf{r}\,\mathrm{dV} = \mathbf{0} \tag{1}$$

holds for all $\Omega \subset \mathcal{D}$. The flux tensor is denoted by $\mathbf{F}$ and $\mathbf{r}$ is the source term. Show that a jump condition across Γ satisfies the condition

$$\mathbf{F}^+.\mathbf{n}^+ = -\mathbf{F}^-.\mathbf{n}^- \quad \text{which is equivalent to} \tag{2a}$$

$$[\![\mathbf{F}]\!]|_\Gamma.\mathbf{n}^+ = (\mathbf{F}^+ - \mathbf{F}^-).\mathbf{n}^+ = \mathbf{0} \tag{2b}$$

for all $\mathbf{x}_0 \in \Gamma$. We assume that the flux $\mathbf{F}$ is continuus on both sides of Γ. [1] (30 points)

Hint: To prove this equation use the balance law for three domains: Ω^+, Ω^-, and $\Omega = \Omega^+ \cup \Omega^-$. By subtracting the balance equations on Ω^+ and Ω^- from that on Ω obtain an integral equation solely on $\Gamma \cap \partial\Omega^+$. Then, "localize" the integral to an arbitrary point $\mathbf{x}_0$ to obtain the point-wise condition.

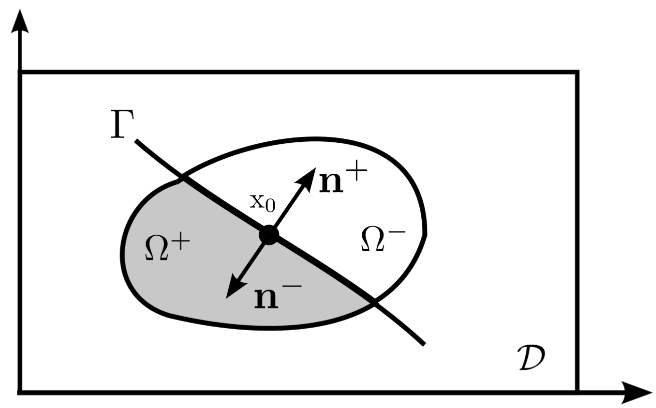


Figure 1: Derivation of the jump conditions from the balance law

2. For elastostatics $\mathbf{F}$ in equation (1) is $-\sigma$ and $\mathbf{r} = \rho\mathbf{b}$, where σ is the stress tensor and $\mathbf{b}$ is the body force per unit mass. (30 points)

   (a) Show that the jump condition for elastostatics across a jump manifold (equation (2a)) reduces to

   $$\mathbf{t}^+ = -\mathbf{t}^- \quad \text{(Newton's second law for tractions).} \tag{3}$$

   Hint: Note that traction is given by $\mathbf{t} = \sigma.\mathbf{n}$ where for traction on each side, σ and $\mathbf{n}$ are taken from the same side.

   (b) Figure 2 shows an example of jump in flux tensor $-\sigma$ for elastostatics. The top surface is uniformly moved up to generate a uniform strain $\epsilon_{22} = \bar{\epsilon}$ in both materials. Since they possess different elastic moduli E^+ and E^-, the stress tensor has a jump across the material interface Γ. Using the solutions provided in the figure, show that in spite of this jump the tractions still satisfy the common action-reaction law at the interface (*cf.* equation (3)).

   (c) For this 2D problem, the strong form is:

$$\sigma_{11,1} + \sigma_{12,2} + \rho b_1 = 0$$
$$\sigma_{21,1} + \sigma_{22,2} + \rho b_2 = 0$$

[1] While physically jumps can occur in $\mathbf{F}$, the strong form of the problem, *i.e.*, $\nabla.\mathbf{F} - \mathbf{r} = 0$, may not be valid at the points on the jump manifold as the (partial) derivatives in $\mathbf{F}$ may not exist.

Discuss whether any of the partial derivatives in this equation cannot be computed across the jump, given that the stress tensor is discontinuous.

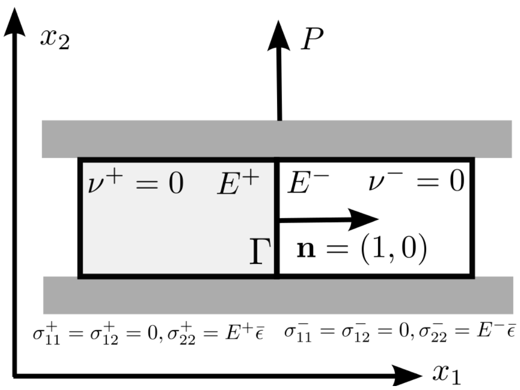


Figure 2: Jump in stress tensor across a material interface.

3. Figure 3(a) shows a possible jump manifold in spacetime for 1D setting. In this case the stress matrix and linear momentum $\mathbf{p} = \rho\mathbf{v}$ reduce to only one component of σ_{11} and p_1, respectively. For brevity, we drop the these indices. As mentioned, the flux matrix for elastodynamics in spacetime is $\mathbf{F} = [-\sigma \ p]$. (30 points)

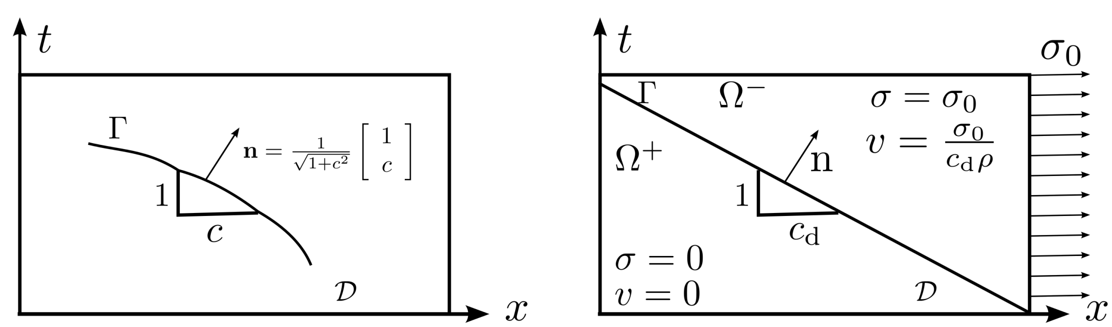
   (a) Given the equation (2b), write the jump conditions for this problem based on the solutions and spacetime normal vector [2] given in figure 3(a).

   (b) Figure 3(b) shows the exact solution to a 1D wave propagation problem where the right boundary of a semi-infinite bar is loaded by stress σ_0. Here $c_{\mathrm{d}} = \sqrt{E/\rho}$ is the material dilatational wave speed. Verity that the jump from the two sides + and − satisfy the jump condition obtained in previous step.

   (c) The strong form of this problem is,

$$\sigma_{,x} + \rho b = \dot{p}$$

   Given the exact solution in the figure, discuss the validity of the strong form for the points on $\mathit{\Gamma}$. Compare this with the elastostatics case in problem 2.c.



(a) Spacetime normal vector $\mathbf{n}$ for a jump manifold moving with speed $-c$.

(b) Exact solution for a problem with applied stress

Figure 3: Jump conditions for elastodynamics

[2] As can be seen, in the calculation of the normal vector, we are adding the square of the speed of the manifold $\mathit{\Gamma}$, c^2, to unity. While the problem can be fixed by using more advanced notations, we do not pursue it further in this course.