1. (**30 Points**) Exercise 77 (Transport theorem for vector fields). You cannot use Transport theorem for scalars (Theorem 145) as proven (still can follow a similar line of proof to show the first identity).

2. (**30 Points**) Exercise 83.

3. (**10 Points**) Exercise 89.

4. (**30 Points**) Exercise 96.

5. (**20 + 40 + 40 = 100 Points**) Transfer between rate, gradient, divergence of quantities in referential and temporal coordinate systems.

   Consider the n'th order tensor field $\mathbf{T}$,

$$\mathbf{T} = T_{i_1\cdots i_n}\mathbf{e}_{i_1} \otimes \cdots \mathbf{e}_{i_n} \tag{1}$$

   We want to show the following,

$$\frac{\mathrm{D}\mathbf{T}}{\mathrm{D}t} = \frac{\partial \hat{\mathbf{T}}}{\partial t} + \mathrm{grad}\hat{\mathbf{T}}\hat{\mathbf{v}} \qquad \left.\frac{\mathrm{D}T_{i_1\cdots i_n}}{\mathrm{D}t}\right|_{\mathbf{x}} = \left.\frac{\partial \hat{T}_{i_1\cdots i_n}}{\partial t}\right|_{\mathbf{y}} + \frac{\partial \hat{T}_{i_1\cdots i_n}}{\partial y_k}\hat{v}_k \tag{2a}$$

$$\mathrm{Grad}\mathbf{T} = \mathrm{grad}\hat{\mathbf{T}}.\mathbf{F} \qquad \frac{\partial T_{i_1\cdots i_n}}{\partial x_j} = \frac{\partial \hat{T}_{i_1\cdots i_n}}{\partial y_k}F_{kj} \tag{2b}$$

$$\mathrm{Div}\mathbf{T} = J\mathrm{div}(\hat{\mathbf{T}}.\mathbf{F}^{\mathrm{T}}/J) \qquad \frac{\partial T_{i_1\cdots i_n}}{\partial x_{i_n}} = J\frac{\partial(\hat{T}_{i_1\cdots i_n}F_{ji_n}/J)}{\partial y_j} \tag{2c}$$

   Note that Div and Grad equations can also be written as,

$$\mathrm{grad}\hat{\mathbf{T}} = \mathrm{Grad}\mathbf{T}.\mathbf{F}^{-1} \qquad \frac{\partial \hat{T}_{i_1\cdots i_n}}{\partial y_k} = \frac{\partial T_{i_1\cdots i_n}}{\partial x_j}F^{-1}_{jk} \tag{3a}$$

$$\mathrm{Div}(J\mathbf{T}\mathbf{F}^{-\mathrm{T}}) = J\mathrm{div}\hat{\mathbf{T}} \qquad \frac{\partial JT_{i_1\cdots i_n}F^{-1}_{ji_n}}{\partial x_j} = J\frac{\partial \hat{T}_{i_1\cdots i_n}}{\partial y_{i_n}} \tag{3b}$$

   We have proven all relations in the class, possibly with different approaches than in this homework assignment.

   (a) Grad: Show (2b) holds.

   (b) Div ($n \geq 1$): Show that,

$$\begin{aligned}\mathrm{Div}(J\mathbf{T}\mathbf{F}^{-\mathrm{T}}) &= J\mathrm{Grad}\mathbf{T} : \mathbf{F}^{-1} + \mathbf{T}.\mathrm{Div}(J\mathbf{F}^{-\mathrm{T}}) \\ &= J\mathrm{div}\hat{\mathbf{T}} + \mathbf{T}.\mathrm{Div}(J\mathbf{F}^{-T})\end{aligned} \tag{4}$$

   The products . and : (not to be confused with 2nd order tensor inner product) are used to denote one and two level contractions. For example:

$$\mathrm{Grad}\mathbf{T} : \mathbf{F}^{-1} = \frac{\partial T_{i_1\cdots i_n}}{\partial x_j}F^{-1}_{ji_n}\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_{n-1}}$$

$$\mathbf{T}.\mathrm{Div}(J\mathbf{F}^{-\mathrm{T}}) = T_{i_1\cdots i_n}\left(\frac{\partial JF^{-1}_{ji_n}}{\partial x_j}\right)\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_{n-1}}$$

   You first need to prove line one of (4) by using product rule for derivatives. Group terms as $\mathbf{T}$ and $J\mathbf{F}^{-\mathrm{T}}$. Then prove line two by using (2b) and showing $(\mathrm{grad}\hat{\mathbf{T}}.\mathbf{F}) : \mathbf{F}^{-1} = \mathrm{div}\hat{\mathbf{T}}$.

(c) Show

$$\mathrm{Div}(J\mathbf{F}^{-T}) = 0 \tag{5}$$

Hint: Use,

$$F^{-1}_{pk} = \frac{1}{2\det\mathbf{F}}\epsilon_{ijk}\epsilon_{mnp}F_{im}F_{jn} \quad \Rightarrow \frac{\partial J F^{-\mathrm{T}}_{kp}}{\partial x_p} = \frac{1}{2}\epsilon_{ijk}\epsilon_{mnp}\frac{\partial F_{im}F_{jn}}{\partial x_p} \tag{6}$$

then use antisymmetry of permutation symbol and symmetry of second derivatives to show this term is zero.

(d) (**0 Points**) By previous two results (4), (5) we have proved (3b). Once can easily prove (2c) from (3b) by appropriate choice of $\mathbf{T}$. This completes the proof for div. We often deal with div operations of the form (3b) rather than (2c); *e.g.,* Piola-Krichhoff stress tensors.

Exercise problems for your practice (DO NOT need to return them)

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1. Calculation of traction from stress tensor: Example 2-1 Saouma (page 40/263).
2. Stress transformation (change of coordinate): Example 2-3 Saouma (page 40/263).
3. Abeyaratne vol II Problem 4.15 on Cauchy stress and traction vector.

Reading assignment: Detailed exposure to topics discussed or those not covered in the class

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1. **balance laws**:
   (a) **Transport equation**: Section "3.7 Transport Equations" from Abyaratne vol II, particularly equations (3.84), (3.85), (3.86), (3.89).
   (b) **Density of physical fields** and path to formulating a balance law: Useful reference Abeyaratne Vol II, 1.8 Extensive Properties and their Densities.
   (c) **Formulation of balance laws**
   - Chapter 4 "Mechanical Balance Laws and Field Equations" (particularly equation 4.2).
   - Section 6.1.1. (balance laws) & 6.1.2 (fluxes) Saouma.

   (d) **Direct expression of balance laws in space and time** (from my FEM course): http://rezaabedi.com/wp-content/uploads/Courses/ContinuumMechanics/BalanceLaws.pdf
   (e) **Jump conditions**: Useful resources are
   - Section "6 Singular Surfaces and Jump Conditions" of Abeyaratne vol II is a good resourse for this topic.
2. **Kinetics**: Useful resources are:
   (a) **Reference configurations and linearization**: Sections "4.8 Formulation of Mechanical Principles with Respect to a Reference Configuration" and "4.10 Linearization" from Abyaratne vol. II.
   (b) **Stress power** Section "4.9 Stress Power" from Abyaratne.
   (c) Calculation of traction from stress tensor: Example 2-1 Saouma (page 40/263).