For linear elastic solid, under small deformation, cauchy stress $\mathbf{T}$ and strain $\mathbf{E}$ are related by,

$$\mathbf{T} = \mathbf{C}\mathbf{E} \qquad \Rightarrow \qquad T_{ij} = C_{ijkl}E_{kl} \tag{1}$$

where for the assumed hyperelastic material $\mathbf{C}$ satisfies major and minor symmetries $C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$.

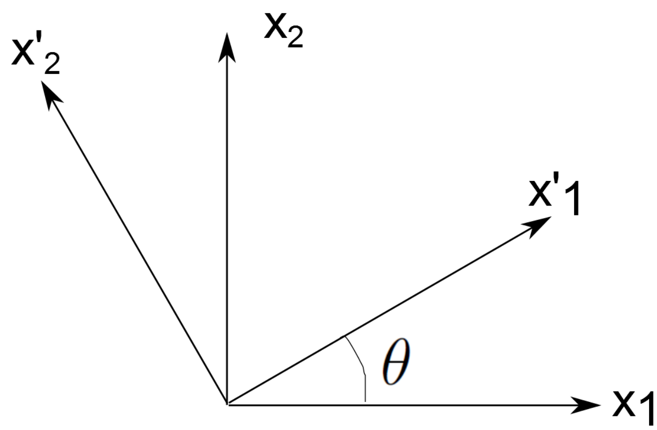

Figure 1: Two coordinate systems $\{e_1, e_2\}$ and $\{e'_1, e'_2\}$ at relative angle θ.

Coordinate transformation from (x, y) to $(x'_1, x_2\prime)$ coordinate system in 2D stipulates,

$$C'_{ijkl} = Q_{im}Q_{jn}Q_{kp}Q_{lq}C_{mnpq} \tag{2}$$

where for the coordinate systems shown we have,

$$\mathbf{Q} = \begin{bmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \qquad \text{for} \quad c = \cos(\theta), s = \sin(\theta) \tag{3}$$

1. (**40 Points**) Express C'_{1111}, C'_{1122}, C'_{1112}, C'_{2222}, C'_{2221}, and C'_{1212} in terms of C_{1111}, C_{1122}, C_{1112}, C_{2222}, C_{2221}, and C_{1212}.

   Hint: Use the major and minor symmetries of $\mathbf{C}$ and (2). For example, for C'_{1111} we have,

$$C'_{1111} = c^4C_{1111} + 2c^2s^2C_{1122} + 4c^3sC_{1112} + s^4C_{2222} + 4cs^3C_{2221} + 4c^2s^2C_{1212} \tag{4}$$

   Express C'_{1122}, C'_{1112} (and ideally the remaining 3) independent components of $[\mathbf{C}']$ in terms of 6 independent components of $[\mathbf{C}]$.

2. (**20 Points**) Voigt stress and strain arrays in 2D are expressed as,

$$\sigma = \begin{bmatrix} T_{11} \\ T_{22} \\ T_{12} \end{bmatrix}, \qquad \gamma = \begin{bmatrix} E_{11} \\ E_{22} \\ 2E_{12} \end{bmatrix} \tag{5}$$

   The Voigt stiffness matrix in 2D is,

$$\sigma = \mathbf{S}\gamma \qquad \text{that is,} \qquad \begin{bmatrix} T_{11} \\ T_{22} \\ T_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ 2E_{12} \end{bmatrix} \tag{6}$$

Show that,

$$S_{11} = C_{1111} \tag{7a}$$
$$S_{12} = S_{21} = C_{1122} \tag{7b}$$
$$S_{13} = S_{31} = C_{1112} \tag{7c}$$
$$S_{22} = C_{2222} \tag{7d}$$
$$S_{23} = S_{32} = C_{2221} \tag{7e}$$
$$S_{33} = C_{1212} \tag{7f}$$

You don't need to show the symmetry of $\mathbf{S}$. That is, you just need to show $S_{12} = C_{1122}$ instead of $S_{12} = S_{21} = C_{1122}$ and the same for the other off diagonals. It suffices to show identities (7)(a,b,c,f).

Hint: Expand (1) in 2D and relate it to (6) to show (7).

3. (**30 Points**) Show that Components of Voigt stiffness in $(x_1', x_2\prime)$ can be written in terms of its components in (x_1, x_2) as shown below (it is sufficient to only show identities for S_{11}' and S_{12}'),

$$S_{11}' = c^4 S_{11} + 2c^2 s^2 S_{12} + 4c^3 s S_{13} + s^4 S_{22} + 4cs^3 S_{23} + 4c^2 s^2 S_{33} \tag{8a}$$
$$S_{12}' = c^2 s^2 S_{11} + (c^4 + s^4) S_{12} - 2cs(c^2 - s^2) S_{13} + c^2 s^2 S_{22} + 2cs(c^2 - s^2) S_{23} - 4c^2 s^2 S_{33} \tag{8b}$$
$$S_{13}' = -c^3 s S_{11} + cs(c^2 - s^2) S_{12} + c^2(c^2 - 3s^2) S_{13} + cs^3 S_{22} + s^2(3c^2 - s^2) S_{23} + 2cs(c^2 - s^2) S_{33} \tag{8c}$$
$$S_{11}' = s^4 S_{11} + 2c^2 s^2 S_{12} - 4cs^3 S_{13} + c^4 S_{22} - 4c^3 s S_{23} + 4c^2 s^2 S_{33} \tag{8d}$$
$$S_{23}' = -cs^3 S_{11} - cs(c^2 - s^2) S_{12} + s^2(3c^2 - s^2) S_{13} + c^3 s S_{22} + c^2(c^2 - 3s^2) S_{23} - 2cs(c^2 - s^2) S_{33} \tag{8e}$$
$$S_{33}' = c^2 s^2 S_{11} - 2c^2 s^2 S_{12} - 2cs(c^2 - s^2) S_{13} + c^2 s^2 S_{22} + 2cs(c^2 - s^2) S_{23} + (c^2 - s^2)^2 S_{33} \tag{8f}$$

where

$$[\mathbf{S}'] = R_\theta(S) = \begin{bmatrix} S_{11}' & S_{12}' & S_{13}' \\ S_{12}' & S_{22}' & S_{23}' \\ S_{13}' & S_{23}' & S_{33}' \end{bmatrix} \tag{9}$$

the notation $R_\theta(S)$ is used later.

Hint: Use (4) (and other 5 similar equations from that problem) and (7).

4. **Cubic material**: Cubic materials are special orthotropic materials that are invariant with respect to 90° rotations with respect to the principal axes. In 2D elasticity, this means that under 90° rotations, the components of constitutive equation do not change. That for $\theta = 90°$, $S_{ab}' = S_{ab}$ for $a, b \in \{1, 2, 3\}$.

   (a) (**30 Points**) From (8) show that for the material to be cubic we need to have,

   $$S_{22} = S_{11} \tag{10a}$$
   $$S_{23} = -S_{13} \tag{10b}$$

   that is, instead of 6 independent components S has only 4 independent components (S_{11}, S_{12}, S_{13}, S_{33}).

   (b) (**30 Points**) **(extra credit)** Principal axes: Show that if $\theta = \theta_p$ the shear-normal coupling terms of stiffness matrix are zero. That is, for

   $$\tan 4\theta_p = \frac{4S_{13}}{S_{11} - S_{12} - 2S_{33}} \tag{11}$$

the expression of $\mathbf{S}$ in (x'_1, x'_2) system is ($S'_{13} = S'_{23} = 0$),

$$[\mathbf{S}'] = \begin{bmatrix} S'_{11} & S'_{12} & 0 \\ S'_{12} & S'_{11} & 0 \\ 0 & 0 & S'_{33} \end{bmatrix} \tag{12}$$

5. (**30 Points**) **Isotropy**: An isotropic elastic material is one that for any angle of rotation the constitutive equation does not change. That is, for any θ the components of $\mathbf{S}'$ are equal to components of $\mathbf{S}$. Clearly, an isotropic material is cubic (as the constitutive relation remains invariant with respect to 90° angle rotations). So, we can start by using (10) (starting with $S_{22} = S_{11}$, S_{12}, $S_{23} = -S_{13}$, and S_{33}). Show that for isotropic material, we can further show,

$$S_{33} = \frac{S_{11} - S_{12}}{2} \tag{13a}$$

$$S_{23} = S_{13} = 0 \tag{13b}$$

Thus, the stiffness is,

$$\begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{11} & 0 \\ 0 & 0 & S_{33} \end{bmatrix}, \quad \text{for} \quad S_{33} = \frac{S_{11} - S_{12}}{2} \tag{14}$$

Note that *Zener* index $Z = \frac{2S_{33}}{S_{11} - S_{12}} = 1$ for isotropic materials.[1]

Hint: Starting from 4 independent components as discussed above, refer to (8) and use $\theta = 45°$ and match the following identities $S'_{33} = S_{33}$ and $S'_{13} = S_{13}$. You do not need to show that other components will remain unchanged for 45° rotation and in fact all components remaining unchanged for any other arbitrary rotation.

6. (**30 Points**) In many crystalline materials there are one or multiple forms of grain with arbitrary angles for the principal axes. If there is no angular bias in principal axes of these grains, the macroscopic material is isotropic. If the grains all have the same (anisotropic) stiffness but with different orientations (which is uniformly distributed in angle) the isotropic limit of stiffness can be expressed as[2],

$$S^{\text{iso}} = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} R_\theta(S) \mathrm{d}\theta \tag{15}$$

where $R_\theta(S)$ would be the stiffness of the base material if rotated by angle $-\theta$. Show that, the components of isotropic stiffness are,

$$S_{11}^{\text{iso}} = S_{22}^{\text{iso}} = \frac{3}{8}(S_{11} + S_{22}) + \frac{1}{4}S_{12} + \frac{1}{2}S_{33} \tag{16a}$$

$$S_{12}^{\text{iso}} = S_{21}^{\text{iso}} = \frac{1}{8}(S_{11} + S_{22}) + \frac{3}{4}S_{12} - \frac{1}{2}S_{33} \tag{16b}$$

$$S_{33}^{\text{iso}} = \frac{1}{8}(S_{11} + S_{22}) - \frac{1}{4}S_{12} + \frac{1}{2}S_{33} \tag{16c}$$

$$S_{13}^{\text{iso}} = S_{31}^{\text{iso}} = S_{23}^{\text{iso}} = S_{32}^{\text{iso}} = 0 \tag{16d}$$

You just need to show the identity for S_{11}^{iso} and S_{12}^{iso}. Note that $\mathbf{S}^{\text{iso}}$ is clearly isotropic with Zener index 1.

7. (**50 Points**) Numerical calculation 1:

[1] Zener anisotropy index is defined for cubic materials, which as shown in (12) has 3 independent parameters once represented in its principal axes. Zener index expresses how anisotropic a cubic material is.

[2] There are more details about this that are not discussed here.

(a) The components of $\mathbf{S}$ in (x_1, x_2) coordinate system are,

$$[S] = \begin{bmatrix} 1.412500000000000 & 0.087500000000000 & -0.238156986040721 \\ 0.087500000000000 & 1.412500000000000 & 0.238156986040721 \\ -0.238156986040721 & 0.238156986040721 & 0.387500000000000 \end{bmatrix} \quad (17)$$

It this material cubic? If so, express for what θ_p it is expressed in its principal form (so that components $S'_{13} = S'_{23} = 0$). Express the components of $\mathbf{S}$ for (x'_1, x'_2) that would correspond to the principal axes and compute the Zener's index (if applicable).

(b) Express the components (17) in (x'_1, x'_2) coordinate system for $\theta = 30°$. If it is cubic, express the Zener index for it. What is the relation of this Zener index and that from the previous case (if this material is cubic).

8. Numerical calculation 2: Consider the stiffness below,

$$[S] = \begin{bmatrix} 1.5 & 0.5 & 0.3 \\ 0.5 & 1 & 0.1 \\ 0.3 & 0.1 & 0.8 \end{bmatrix} \quad (18)$$

Compute (**20 Points**) $\mathbf{S}^{\text{iso}}$.

Reading assignment: Detailed exposure to topics discussed or those not covered in the class

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1. **Energy related material**:
   (a) **Virtual work**: Abeyaratne Vol II, Problem 4.18, page 143.
   (b) **Energy conjugates**: Abeyaratne Vol II, Problem 4.20, 4.22, 4.23: pages 143-145.
2. **Constitutive equation**:
   (a) **Nonlinear isotropic solid**: Abeyaratne Vol II, 8.5.2.1, page 238.
   (b) **Compressive fluid**: Abeyaratne Vol II, 8.7.1, page 223.
   (c) **Popular nonlinear solid constitutive models**: Abeyaratne Vol II, 8.7.2 (neo-Hookean), 8.7.4 (Incompressible isotropic solid).
   (d) 6×6 **stiffness matrix (Voigt notation)**: This 6×6 stiffness matrix is easier to use than $3 \times 3 \times 3 \times 3$ elasticity tensor. Refer to section 2 (pages 10-15) of http://rezaabedi.com/wp-content/uploads/2014/04/Elastostatics.pdf
   (e) **Various anisotropy models for linear elasticity**: Saouma 7.3.1 Anisotropy, 7.3.2 Monotropic, 7.3.3 Orthotropic, 7.3.4 Transversely Isotropic, 7.3.5 Isotropic.